THE ULAM-HYERS STABILITY OF TWO-VARIABLE RADICAL FUNCTIONAL EQUATIONS IN QUASI-BANACH SPACES

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Abstract

The purpose of this study is to prove Ulam-Hyers stability of two-variable radical functional equations in quasi-Banach spaces. As a consequence of the main result, we get an outcome on the stability of such functional equations in Banach spaces.

Keywords: *Quasi-Banach space, radical functional equation, Ulam-Hyers stability.*

TÍNH ỔN ĐỊNH ULAM-HYERS CỦA PHƯƠNG TRÌNH HẦM HAI BIẾN TRÊN KHÔNG GIAN TỰA BANACH

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Tóm tắt

Mục đích của bài báo là chứng minh tính ổn định Ulam-Hyers của phương trình hàm căn 2 biến trên không gian tựa Banach. Hệ quả thu được là tính ổn định Ulam-Hyers của phương trình hàm này trên không gian Banach.

Từ khóa: *Không gian tựa Banach, ổn định Ulam-Hyers, phương trình hàm căn.*

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1. Introduction

The first stability problem concerning the group homomorphisms was introduced by Ulam (Ulam, 1960) and the first partial result on stability for additive mapping in Banach spaces was proposed by Hyers (Hyers, 1941). Later, the stability of functional equations has been studied in many different types of spaces. Recently, studying the stability of functional equations in quasi-Banach spaces has attracted many authors (Nguyen & Vo, 2018; Nguyen & Nguyen, 2021; Nguyen & Sintunavarat, 2019; Eskandani, 2008; Najati & Moghimi, 2008).

The quasi-normed space is a generalization of a normed space (Kalton, 2003; Kalton et al., 1984). The difference between a quasi-norm and a norm is the modulus of concavity $\kappa \geq 1$, see Definition 1.1.(3) below. As a consequence, a quasi-norm is not necessarily continuous and the inequality does not necessarily hold for more than two points.

In 2021, Iz-Iddine El-Fassi introduced a new type of two-variable radical functional equations as follows:

$$f(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) = f(x,y)+f(u,v) \qquad (1)$$

for all $x, y, u, v \in \mathbb{R}$ and $k, l \in \mathbb{N}$ are fixed numbers. Then, the authors also studied the generalized hyperstability of such an equation.

In this paper, we investigate the Ulam-Hyers stability of two-variable radical functional equations in quasi-Banach spaces. We also deduce a consequence of the stability of such functional equations in Banach spaces.

First, we recall some results on quasi-Banach space (Aoki, 1942; Kalton et al., 1984), which is useful in the main results.

1.1. Definition

Let X be a vector space over the field $\mathbb{K}$, $\kappa \geq 1$ and $\|.\|: X \to \mathbb{R}_+$ be a function such that for all $x, y \in X, r \in \mathbb{K}$,

1. $\|x\| = 0$ if and only if $x = 0$.
2. $\|rx\| = |r|.\|x\|$.
3. $\|x+y\| \leq \kappa(\|x\|+\|y\|)$.

Then,

1. $\|.\|$ is called a *quasi-norm* in X, the possible smallest κ is called the *modulus of concavity* and $(X, \|.\|, \kappa)$ is called a *quasi-normed space*. For a quasi-normed space $(X, \|.\|, \kappa)$, without loss of the generality we can assume that κ is the modulus of concavity.

2. $\|.\|$ is called a p-*norm* and $(X, \|.\|, \kappa)$ is called a p-*normed space* if for some $p \in (0;1]$ and all $x, y \in X$, $\|x+y\|^p \leq \|x\|^p + \|y\|^p$. (2)

3. The sequence $\{x_n\}$ is called *convergent* to x if $\lim_{n\to\infty} \|x_n - x\| = 0$, written $\lim_{n\to\infty} x_n = x$.

4. The sequence $\{x_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} \|x_n - x_m\| = 0$.

5. The quasi-normed space $(X, \|.\|, \kappa)$ is called a *quasi-Banach space* if each Cauchy sequence is convergent.

6. The quasi-normed space $(X, \|.\|, \kappa)$ is called a p-*Banach space* if it is p-normed and quasi-Banach.

The next theorem show that each quasi-norm is equivalent to some *p*-norm (Maligranda, 2008). This is an important tool to prove the main results. It is called *Aoki–Rolewicz theorem.*

1.2. Theorem (*Aoki–Rolewicz theorem*).

Let $(X, \kappa, \|.\|)$ be a quasi-normed space, $p = \log_{2\kappa} 2$ and

$$|||x||| = \inf\{(\sum_{i=1}^{n} \|x_i\|^p)^{\frac{1}{p}} : x = \sum_{i=1}^{n} x_i, x_i \in X, n \geq 1\}$$

for all $x \in X$. Then, $|||.|||$ is a quasi-norm on X satisfying $|||x+y|||^p \leq |||x|||^p + |||y|||^p$ (3)

and $\frac{1}{2\kappa}\|x\| \leq |||x||| \leq \|x\|$ (4)

for all $x, y \in X$. In particular, the quasi-norm $|||.|||$ is a p-norm and if $\|.\|$ is a norm then $p = 1$ and $|||.||| = \|.\|$.

2. Main results

First, we show a property of the two-variable radical functional equation (1).

2.1. Lemma

Let X be a vector space and $F:\mathbb{R}^2 \to X$ be a mapping satisfying (1). Then, F satisfies the following equation $F(x,y)=2^{-n}F(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)$ (5) for all $x,y\in\mathbb{R}$ and all $n\in\mathbb{N}$.

Proof. By replacing $u=x,v=y$ in (1), we find that

$$0=F(\sqrt[k]{x^k+x^k},\sqrt[l]{y^l+y^l})-F(x,y)-F(x,y)$$

$$=F(\sqrt[k]{2}x,\sqrt[l]{2}y)-2F(x,y).$$

This implies that $F(\sqrt[k]{2}x,\sqrt[l]{2}y)=2F(x,y)$ for all $x,y\in\mathbb{R}$. It follows that the formula (5) holds for $n=1$. Suppose that (5) holds for some positive integer $n>1$, that is $F(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)=2^nF(x,y)$ for all $x,y\in\mathbb{R}$.

For all $x,y\in\mathbb{R}$, we find that

$$F(\sqrt[k]{2^{n+1}}x,\sqrt[l]{2^{n+1}}y)=F(\sqrt[k]{2}.\sqrt[k]{2^n}x,\sqrt[l]{2}.\sqrt[l]{2^n}y)$$

$$=2F(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)\ =2.2^nF(x,y)\ =2^{n+1}F(x,y).$$

Therefore, the formula (5) holds.

Finally, we investigate the stability of the two-variable radical functional equation (1) in quasi-Banach spaces.

2.2. Theorem

Let $(X,\kappa,\|.\|)$ be a quasi-Banach space, $f:\mathbb{R}^2\to X$ be a mapping and $\varepsilon>0$ satisfying

$$\|f(\sqrt[k]{x^k+u^k},\sqrt[l]{y^l+v^l})-f(x,y)-f(u,v)\|\le\varepsilon \quad (6)$$

for all $x,y,u,v\in\mathbb{R}$ and $k,l\in\mathbb{N}$.

Then, there exists a unique mapping $F:\mathbb{R}^2\to X$ satisfying the following

1. F is a solution of the two-variable radical functional equation (1).

2. $F(x,y)=\lim_{n\to\infty}2^{-n}f(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)$ for all $x,y\in\mathbb{R}$.

3. For all $x,y\in\mathbb{R}$, $p=\log_{2\kappa}2$,

$$\|f(x,y)-F(x,y)\|\le\left(\frac{2}{2^p-1}\right)^{\frac{1}{p}}\varepsilon. \quad (7)$$

Proof. By replacing $u=x,v=y$ in inequality (6), we find that

$$\|2^{-1}f(\sqrt[k]{2}x,\sqrt[l]{2}y)-f(x,y)\|\le\frac{\varepsilon}{2}. \quad (8)$$

Applying Theorem 1.2 to (8), we obtain

$$|||2^{-1}f((\sqrt[k]{2}x,\sqrt[l]{2}y)-f(x,y)|||^p$$

$$\le\|2^{-1}f((\sqrt[k]{2}x,\sqrt[l]{2}y)-f(x,y)\|^p\le\frac{\varepsilon^p}{2^p}. \quad (9)$$

Set $F_n(x,y)=2^{-n}f(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)$ for all $x,y\in\mathbb{R}$. By induction, we prove that

$$|||f(x,y)-F_n(x,y)|||^p\le\frac{1-2^{-np}}{2^p-1}\varepsilon^p \quad (10)$$

for all $x,y\in\mathbb{R}$, $n\in\mathbb{N}$. It follows from (9) that (10) holds for $n=1$. Assume that the formula (10) holds for some positive integer $n>1$. Then, using the assumption of induction and (9), we obtain

$$|||f(x,y)-F_{n+1}(x,y)|||^p$$

$$\le|||f(x,y)-F_n(x,y)|||^p+|||F_n(x,y)-F_{n+1}(x,y)|||^p$$

$$\le\frac{1-2^{-np}}{2^p-1}\varepsilon^p+|||2^{-n}f(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)-2^{-(n+1)}f(\sqrt[k]{2^{n+1}}x,\sqrt[l]{2^{n+1}}y)|||^p$$

$$\le\frac{1-2^{-np}}{2^p-1}\varepsilon^p+2^{-np}|||f(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)-2^{-1}f(\sqrt[k]{2}\sqrt[k]{2^n}x,\sqrt[l]{2}\sqrt[l]{2^n}y)|||^p$$

$$\le\frac{1-2^{-np}}{2^p-1}\varepsilon^p+\frac{2^{-np}\varepsilon^p}{2^p}$$

$$=\frac{1-2^{-np}}{2^p-1}\varepsilon^p+\frac{2^{-np}(1-2^{-p})}{2^p-1}\varepsilon^p$$

$$=\frac{1-2^{-(n+1)p}}{2^p-1}\varepsilon^p.$$

for all $x,y\in\mathbb{R}$. It follows that the formula (10) holds.

Next, we claim that $\{F_n(x,y)\}$ is a Cauchy sequence in X for all $x,y\in\mathbb{R}$. Letting $m,n\in\mathbb{N}$ and $m>n$, by applying (10) and Theorem 1.2, we obtain

$$\frac{1}{2}\|F_m(x,y)-F_n(x,y)\|^p$$

$$\le|||F_m(x,y)-F_n(x,y)|||^p$$

$$\le|||2^{-m}.f(\sqrt[k]{2^m}x,\sqrt[l]{2^m}y)-2^{-n}.f(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)|||^p$$

$$\le2^{-np}.|||2^{-(m-n)}.f(\sqrt[k]{2^{m-n}}.\sqrt[k]{2^n}x,\sqrt[l]{2^{m-n}}.\sqrt[l]{2^n}y)-f(\sqrt[k]{2^n}x,\sqrt[l]{2^n}y)|||^p$$

$$\le 2^{-np} ||| F_{m-n}(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) |||^p$$

$$\le 2^{-np}.\frac{1-2^{-(m-n)p}}{2^p-1}\varepsilon^p \quad (11)$$

for all $x, y \in \mathbb{R}$. Then, taking the limit as $n, m \to \infty$ in the inequality (11), we obtain

$$\lim_{n,m\to\infty} \| F_m(x,y) - F_n(x,y) \| = 0$$

for all $x, y \in \mathbb{R}$. Thus, we conclude that $\{F_n(x,y)\}$ is a Cauchy sequence in X. Since X is a quasi-Banach, there exists $F: \mathbb{R}^2 \to X$ such that

$$F(x,y) = \lim_{n\to\infty} F_n(x,y) \quad (12)$$

for all $x, y \in \mathbb{R}$. Taking the limit as $n \to \infty$ in (10) and applying Theorem 1.2 and (12), we obtain

$$\frac{1}{2} \| f(x,y) - F(x,y) \|^p \le ||| f(x,y) - F(x,y) |||^p$$

$$= ||| f(x,y) - \lim_{n\to\infty} F_n(x,y) |||^p$$

$$= \lim_{n\to\infty} ||| f(x,y) - F_n(x,y) |||^p$$

$$\le \lim_{n\to\infty} \left(\frac{1-2^{-np}}{2^p-1}\varepsilon^p\right) = \frac{\varepsilon^p}{2^p-1} \quad (13)$$

for all $x, y \in \mathbb{R}$. Then, approximation (7) is a consequence of (13).

Now, we prove F is a solution of the two-variable radical functional equation (1). By applying Theorem 1.2 and the assumption (6), we find that

$$||| F_n(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - F_n(x,y) - F_n(u,v) |||$$

$$\le \| F_n(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - F_n(x,y) - F_n(u,v) \|$$

$$= \| 2^{-n}.f(\sqrt[k]{2^n}.\sqrt[k]{x^k+u^k}, \sqrt[l]{2^n}.\sqrt[l]{y^l+v^l}) - 2^{-n}.f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - 2^{-n}.f(\sqrt[k]{2^n}u, \sqrt[l]{2^n}v) \|$$

$$= 2^{-n} \| f(\sqrt[k]{2^n}.\sqrt[k]{x^k+u^k}, \sqrt[l]{2^n}.\sqrt[l]{y^l+v^l}) - f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - f(\sqrt[k]{2^n}u, \sqrt[l]{2^n}v) \|$$

$$= 2^{-n}.\| f(\sqrt[k]{(\sqrt[k]{2^n}x)^k + (\sqrt[k]{2^n}u)^k}, \sqrt[l]{(\sqrt[l]{2^n}y)^l + (\sqrt[l]{2^n}v)^l}) - f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - f(\sqrt[k]{2^n}u, \sqrt[l]{2^n}v) \|$$

$$\le 2^{-n}\varepsilon \quad (14)$$

for $x, y, u, v \in \mathbb{R}$. Taking the limit as $n \to \infty$ in (14) and using (12), we obtain

$$||| F(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - F(x,y) - F(u,v) |||$$

$$= ||| \lim_{n\to\infty} F_n(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - \lim_{n\to\infty} F_n(x,y) - \lim_{n\to\infty} F_n(u,v) |||$$

$$= \lim_{n\to\infty} ||| F_n(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - F_n(x,y) - F_n(u,v) |||$$

$$\le \lim_{n\to\infty} (2^{-n}\varepsilon) = 0 \quad (15)$$

for all $x, y, u, v \in \mathbb{R}$. It follows from (15) that

$$F(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - F(x,y) - F(u,v) = 0$$

for all $x, y, u, v \in \mathbb{R}$, that means F is a solution of the two-variable radical functional equation (1).

Finally, we prove the uniqueness of the mapping F. Suppose that there exists a mapping $G: \mathbb{R}^2 \to X$ such that G is also a solution of the two-variable radical functional equation (1) and satisfying the following approximation (7). It follows from Lemma 2.1 that

$$F(x,y) = 2^{-n} F(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y),$$

$$G(x,y) = 2^{-n} G(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y)$$

for all $x, y \in \mathbb{R}$. Using Lemma 2.1 and (13), we obtain

$$||| F(x,y) - G(x,y) |||$$

$$= ||| 2^{-n} F(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - 2^{-n} G(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) |||^p$$

$$= 2^{-np}.||| F(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - G(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) |||^p$$

$$\le 2^{-np}.(||| F(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) |||^p$$

$$+ ||| f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) - G(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y) |||^p)$$

$$\le 2^{-np}.\left(\frac{\varepsilon}{2^p-1} + \frac{\varepsilon}{2^p-1}\right) \le \frac{2^{-np+1}\varepsilon}{2^p-1} \quad (16)$$

for all $x, y \in \mathbb{R}$. Taking the limit as $n \to \infty$ in inequality (16), we obtain $F(x,y) = G(x,y)$ for all $x, y \in \mathbb{R}$. Hence, $F = G$ and the theorem has been proved.

2.3. Remark

By choosing $\kappa = 1$ in the Theorem 2.2, we obtain the result on stability of the two-variable radical functional equation (1) in Banach spaces. Note that, in Theorem 2.2 if $\|.\|$ is a norm then $p = 1$ and $|||.||| = \|.\|$.

2.4. Corollary

Let $(X, \|.\|)$ *be a Banach space,* $\varepsilon > 0$ *and* $f: \mathbb{R}^2 \to X$ *be a mapping such that*

$$\| f(\sqrt[k]{x^k+u^k}, \sqrt[l]{y^l+v^l}) - f(x,y) - f(u,v) \| \le \varepsilon$$

for all $x, y, u, v \in \mathbb{R}$ *and* $k, l \in \mathbb{N}$.

Then, there exists a unique mapping $F : \mathbb{R}^2 \to X$ *satisfying the following*

1. F *satisfies the two-variable radical mapping equation (1).*

2. $F(x, y) = \lim_{n\to\infty} 2^{-n} f(\sqrt[k]{2^n}x, \sqrt[l]{2^n}y)$ *for all* $x, y \in \mathbb{R}$.

3. F *satisfies the approximation* $\| f(x, y) - F(x, y) \| \leq \varepsilon$ *for all* $x, y \in \mathbb{R}$.

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