SUBDIFFERENTIALS WITH DEGREES OF FREEDOM AND APPLICATIONS TO OPTIMIZATION PROBLEMS

Vo Duc Thinh* **and Huynh Ngoc Cam**

Faculty of Mathematics - Informatics Teacher Education, School of Education,

Dong Thap University, Cao Lanh 87000, Vietnam

Corresponding author: Vo Duc Thinh, Email: vdthinh@dthu.edu.vn

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Abstract

In this work, we first present a new class of generalized differentials, namely subdifferentials with degrees of freedom as well as their applications in nonsmooth optimization problems. We then establish some computation rules for subdifferentials with degree of frecdom of functions under basic qualification constraints. By using these computation rules, we provide necessary and sufficient conditions for unconstraint optmization problems and for optimization problems with geometric constraints.

Keywords: *Computation rule, generalized convex function, subdifferential, optimality condition.*

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DƯỚI VI PHÂN CÓ BẬC TỰ DO VÀ NHỮNG ÁP DỤNG VÀO BÀI TOÁN TỐI ƯU

Võ Đức Thịnh* **và Huỳnh Ngọc Cảm**

Khoa Sư phạm Toán - Tin, Trường Sư phạm, Trường Đại học Đồng Tháp, Việt Nam

**Tác giả liên hệ: Võ Đức Thịnh, Email: vdthinh@dthu.edu.vn*

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Tóm tắt

Trong bài viết này, trước tiên chúng tôi giới thiệu một lớp mới của các vi phân suy rộng, được gọi là dưới vi phân có bậc tự do, cũng như những áp dụng của chúng vào các bài toán tối ưu không trơn. Sau đó, chúng tôi thiết lập một số quy tắc tính cho dưới vi phân có bậc tự do của những hàm số dưới điều kiện chuẩn hoá cơ bản. Sử dụng những quy tắc tính này, chúng tôi cung cấp điều kiện cần và đủ cho bài toán tối ưu không ràng buộc và bài toán tối ưu với ràng buộc tập.

Từ khóa: *Dưới vi phân, điều kiện tối ưu, hàm lồi tổng quát, quy tắc tính.*

1. Introduction and Preliminaries

General convexity as well as results on the analysis of generalized convex functions have been of interest to many authors (Ansari et al., 2014; Lara et al., 2021; Lara et al., 2022; Lara, 2022; Kagani et al., 2022). Some different types of subdifferentials were introduced in the literature. It was known that different types of subdifferentials are perfectly suited to a different class of functions, such as: the convex subdifferential for convex functions (Rockafellar, 1996), the Clarke's subdifferential for locally Lipschitz continuous functions (Clarke, 1983), and the strong subdifferentials for strongly quasiconvex functions (Lara, 2022; Kabgani et al. (2022). Inspired from Lara et al. (2021, Kagani et al. (2022), and Thinh et al. (2024), we present, in this work, new subdifferentials, namely subdifferentials with degrees of freedom of nonconvex functions, and then we state calculation rules for such subdifferentials. Using obtained calculation rules, we provide the necessary and sufficient conditions for points to be solutions to optimization problems. So, we also propose a class of functions that are perfectly suited to subdifferentials with degrees of freedom. Besides, one way to define subdifferentials of a single-valued mapping was based on normal cones to its epigraph.

In this paper, we first introduce a new generalized normal set as a generalization of convex normal cones. After that, we establish formulas for calculating such normal sets. These formulas are basic for establishing calculation rules for subdifferentials with degrees of freedom of functions belonging to a class of new generalized convex functions.

Throughout this paper, we always assume that X is a Banach space with norm $\|\cdot\|$ and its topological dual X^*. Let C be a nonemptyset in X, we define

$$\delta_C(x) := \begin{cases} 0 & \text{if } x \in C \\ \infty & \text{otherwise.} \end{cases}$$

For a pair $(x, x^*) \in X \times X^*$, the symbol $\langle x^*, x\rangle$ indicates the canonical pairing between X and X^*. Let C be a nonempty subset of X. The relative interior of C defined by

$$\text{re (C)} := \{x \in C \mid \exists \epsilon > 0 : \mathbb{B}_\epsilon(x) \cap \text{aff}\,(C) \subset C\}$$

where $\mathbb{B}_\epsilon(x)$ is a ball of radius ϵ and centered on x, and aff (C) is the affine hull of C. The (convex) normal cone to C at $\bar{x} \in C$ is given by

$N(x, C) := \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \le 0\ \forall x \in C\}$.

Let $f: X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. The singular subdifferential of f at $\bar{x} \in \text{dom } f$ is defined by

$\partial^\infty f(\bar{x}) := N(\bar{x}, \text{dom } f)$.

2. Normal sets with degrees of freedom to sets

Let C be a nonempty closed convex subset of X. The mapping $\nu: C \times C \to \mathbb{R}$ which satisfies (i) $\nu(x, x) = 0$ for all $x \in C$, and (ii) $\lim_{x \to \bar{x}} \nu(x, \bar{x}) \to 0$ for all $\bar{x} \in C$ is called to be a like-distance function on C.

Let $\emptyset \neq \Omega \subset X$ and let $\nu: X \times X \to \mathbb{R}$ be a like-distance function on X. The ν-normal set (or normal set with degree of freedom) to Ω at $\bar{x} \in \Omega$ is defined by

$$N^\nu(\bar{x}, \Omega) := \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle + \nu(x, \bar{x}) \le 0, \forall x \in \Omega\}.$$

If $\nu(\cdot, \bar{x}) \equiv 0$ then $N^\nu(\bar{x}, \Omega)$ reduces to the convex normal cone to Ω at $\bar{x}$.

We now establish formulas for calculating of ν-normal sets to sets. In the the case of $\nu(\cdot\ \bar{x}) = 0$ for all $x \in X$, the following results reduce to the exiting results in convex analysis.

Theorem 1. Let Ω_1, Ω_2 be subsets of X with $\bar{x} \in \Omega_1$ and $\bar{y} \in \Omega_2$. Let $\nu_1, \nu_2: X \times X \to \mathbb{R}$ be like-distance functions. Then we get

$$N^\nu((\bar{x}, \bar{y}), \Omega_1 \times \Omega_2) = N^{\nu_1}(\bar{x}, \Omega_1) \times N^{\nu_2}(\bar{x}, \Omega_2) \quad (1.1)$$

where $\nu: X^2 \times X^2 \to \mathbb{R}$ is defined by $\nu((x, y); (u, v)) := \nu_1(x, u) + \nu_2(y, v)$ for all $(x, y), (u, v) \in X^2$.

Proof. Take $v = (v_1, v_2) \in X \times X$ satisfying $v \in N^\nu((\bar{x}, \bar{y}), \Omega_1 \times \Omega_2)$. By the definition, we get

$\langle (v_1, v_2), (x, y) - (\bar{x}, \bar{y})\rangle + \nu((x, y), (\bar{x}, \bar{y})) \le 0, \forall (x, y) \in \Omega_1 \times \Omega_2$.

This is equivalent to

$$\langle v_1, x - \bar{x}\rangle + \nu_1(x, \bar{x}) + \langle v_2, y - \bar{y}\rangle + \nu_2(y, \bar{y}) \le 0, \forall x \in \Omega_1, y \in \Omega_2. \quad (1.2)$$

Taking $y = \bar{y}$ into account, we get

$$\langle v_1, x - \bar{x}\rangle + \nu_1(x, \bar{x}) \le 0, \forall x \in \Omega_1$$

which means that $v_1 \in N^{\nu_1}(\bar{x}, \Omega_1)$.

Similarly, picking $x = \bar{x}$ in (1.2), we get

$$\langle v_2, y - \bar{y}\rangle + \nu_2(y, \bar{y}) \le 0, \forall y \in \Omega_2$$

which implies that $v_2 \in N^{\nu_2}(\bar{y}, \Omega_2)$.

Thus, we have

$$N^{\nu}((\bar{x},\bar{y}),\Omega_1\times\Omega_2)\subset N^{\nu_1}(\bar{x},\Omega_1)\times N^{\nu_2}(\bar{y},\Omega_2). \quad (1.3)$$

Otherwise, let $v_1\in N^{\nu_1}(\bar{x},\Omega_1)$ and $v_2\in N^{\nu_2}(\bar{y},\Omega_2)$. By the definition, we get

$$\langle v_1,x-\bar{x}\rangle+\nu_1(x,\bar{x})\le 0,\forall x\in\Omega_1$$

and

$$\langle v_2,y-\bar{y}\rangle+\nu_2(y,\bar{y})\le 0,\forall y\in\Omega_2.$$

It follows that

$$\langle v_1,x-\bar{x}\rangle+\nu_1(x,\bar{x})+\langle v_2,y-\bar{y}\rangle+\nu_2(y,\bar{y}) \le 0,\forall x\in\Omega_1,y\in\Omega_2$$

which is equivalent to

$$\langle (v_1,v_2),(x,y)-(\bar{x},\bar{y})\rangle+\nu\big((x,y),(\bar{x},\bar{y})\big) \le 0,\forall (x,y)\in\Omega_1\times\Omega_2.$$

This deduces that

$$(v_1,v_2)\in N^{\nu}((\bar{x},\bar{y}),\Omega_1\times\Omega_2)$$

which gives us the following relation

$$N^{\nu_1}(\bar{x},\Omega)\times N^{\nu_2}(\bar{y},\Omega_2)\subset N^{\nu}\big((\bar{x},\bar{y}),\Omega_1\times\Omega_2\big). \quad (1.4)$$

It implies from (1.3) and (1.4) that (1.1) holds.

Corollary 2. Let Ω_1,Ω_2 be subsets of X with $\bar{x}\in\Omega_1$ and $\bar{y}\in\Omega_2$. Let $\tilde{\nu}: X\times X\to\mathbb{R}$ be a like-distance function. Then, we get

$$N^{\nu}((\bar{x},\bar{y}),\Omega_1\times\Omega_2)=N^{\tilde{\nu}}(\bar{x},\Omega_1)\times N(\bar{x},\Omega_2) \quad (1.5)$$

where $\nu: X^2\times X^2\to\mathbb{R}$ is defined by $\nu((x,y),(u,v)):=\tilde{\nu}(x,u)$ for all $(x,y),(u,v)\in X^2$.

Proof**.** It directly implies from Theorem 1 with $\nu_1(x,u)=\tilde{\nu}(x,u)$ and $\nu_2(y,v)=0$ for any $(x,u),(y,v)\in X^2$.

Theorem 3. Let Ω_1,Ω_2 be convex subsets of X with $\bar{x}\in\Omega_1\cap\Omega_2$. Let $\nu: X\times X\to\mathbb{R}$ be a like-distance function satisfying $\nu(\cdot,\bar{x}): X\to\mathbb{R}$ is concave on Ω_2. Then, for any $v\in N^{\nu}(\bar{x},\Omega_1\cap\Omega_2)$, there exist $\lambda\in\{0,1\}$ and $v_1\in N(\bar{x},\Omega_1), v_2\in N^{\lambda\nu}(\bar{x},\Omega_2)$ such that $(\lambda,v_1)\ne 0$ and

$$\lambda v=v_1+v_2. \quad (1.6)$$

Proof. Take $v\in N^{\nu}(\bar{x},\Omega_1\cap\Omega_2)$. By the definition, we get

$$\langle v,x-\bar{x}\rangle+\nu(x,\bar{x})\le 0,\forall x\in\Omega_1\cap\Omega_2.$$

Set $\Theta_1:=\Omega_1\times[0,\infty)$ and $\Theta_2:=\{(x,\lambda)\in\Omega_2\times\mathbb{R}\mid \lambda\le\langle v,x-\bar{x}\rangle+\nu(x,\bar{x})\}$. By the convexity of Ω_1,Ω_2 and the concavity of $\nu(\cdot,\bar{x})$, it exists that Θ_1,Θ_2 are convex sets. Moreover, Θ_1,Θ_2 form an extremal system due to the fact that for arbitrarily $a>0$, we get $\Theta_1\cap(\Theta_2-(0,a))=\emptyset$.

By using [Mordukhovich et al. (2014), Theorem 2.8], we find $(w,\gamma)\in X\times\mathbb{R}$ separating Θ_1 and Θ_2 in the sense that

$$\langle w,x\rangle+\gamma\lambda_1\le\langle w,y\rangle+\gamma\lambda_2,\forall (x,\lambda_1)\in\Theta_1,\forall (y,\lambda_2)\in\Theta_2. \quad (1.7)$$

It is clear that $\gamma\le 0$. Indeed, if the opposite is that $\gamma>0$, then picking $(x,\lambda_1)=(\bar{x},1)$ and $(y,\lambda_2)=(\bar{y},0)$ in (1.7), we get $\gamma\le 0$. This is a contradiction. Let's consider the following two cases:

Case 1. $\gamma=0$. In this case, $w\ne 0$. Moreover, it implies from (1.7) that

$$\langle w,x\rangle\le\langle w,y\rangle,\forall x\in\Omega_1,y\in\Omega_2$$

which follows that $w\in N(\bar{x},\Omega_1)$ and $-w\in N(\bar{x},\Omega_2)=N^{\gamma\nu}(\bar{x},\Omega_2)$. Thus, the assertion (1.6) holds with $\lambda=0$ and $v_1=w, v_2=-w$.

Case 2. $\gamma<0$. For any $x\in\Omega_1$, taking $(x,\lambda_1)=(x,0)\in\Theta_1$ and $(y,\lambda_2)=(\bar{x},0)\in\Theta_2$ into (1.7), we obtain $\langle w,x-\bar{x}\rangle\le 0,\forall x\in\Omega_1$ which means that $w\in N(\bar{x},\Omega_1)$. So $-\frac{w}{\gamma}\in N(\bar{x},\Omega_1)$. To proceed further, for any $x\in\Omega_2$, picking $(\bar{x},0)\in\Theta_1$ and $(x,\langle v,x-\bar{x}\rangle+\nu(x,\bar{x}))\in\Theta_2$ in (1.7), we get

$$\langle w,x-\bar{x}\rangle+\gamma(\langle v,x-\bar{x}\rangle+\nu(x,\bar{x}))\ge 0. \quad (1.8)$$

Dividing both sides of (1.8) by γ, we get

$$\left\langle\frac{w}{\gamma}+v,x-\bar{x}\right\rangle+\nu(x,\bar{x})\le 0\forall x\in\Omega_2.$$

which gives us that $\frac{w}{\gamma}+v\in N^{\nu}(\bar{x},\Omega_2)$. Thus (1.6) holds with $v_1=\frac{-w}{\gamma}, v_2=\frac{w}{\gamma}+v$ and $\lambda=1$. Hence, the proof is completed.

Theorem 4. Let Ω_1,Ω_2 be nonempty convex subsets of X and $\bar{x}\in\Omega_1\cap\Omega_2$. Let $\nu: X\times X\to\mathbb{R}$ be a like-distance function which satisfies $\nu(\cdot,\bar{x}): X\to\mathbb{R}$ is concave on Ω_2. Assume that the basic qualification condition (BQC) is satisfied

$$N(\bar{x},\Omega_1)\cap[-N(\bar{x},\Omega_2)]=\{0\}.$$

Then we get

$$N^{\nu}(\bar{x},\Omega_1\cap\Omega_2)=N(\bar{x},\Omega_1)+N^{\nu}(\bar{x},\Omega_2).$$

Proof**.** Take $v\in N^{\nu}(\bar{x},\Omega_1\cap\Omega_2)$. By using Proposition 3, we find $\lambda\in\{0,1\}$, $v_1\in N(\bar{x},\Omega_1)$ and $v_2\in N^{\lambda\nu}(\bar{x},\Omega_2)$ such that $(\lambda,v_1)\ne 0$ and $\lambda v=v_1+v_2$. If $\lambda=0$ then $v_1\ne 0$ and $v_2=-v_1\in N(\bar{x},\Omega_2)$. Thus $0\ne v_1\in[-N(\bar{x},\Omega_2)]\cap N(\bar{x},\Omega_1)$ which contradicts to (BQC). So $\lambda=1$. Using Proposition 3 again, we find $v_1\in N(\bar{x},\Omega_1)$ and $v_2\in N^{\nu}(\bar{x},\Omega_2)$ such that $v=v_1+v_2$ which follows that

$$N^{\nu}(\bar{x},\Omega_1\cap\Omega_2)\subset N(\bar{x},\Omega_1)+N^{\nu}(\bar{x},\Omega_2).$$

To show the opposite inclusion, we take $v_1 \in N(\bar{x},\Omega_1)$ and $v_2 \in N^\nu(\bar{x},\Omega_2)$. We have from the definition that
$\langle v_1, x-\bar{x}\rangle \le 0, \forall x \in \Omega_1$ and $\langle v_2, x-\bar{x}\rangle + \nu(x,\bar{x}) \le 0, \forall x \in \Omega_2$.
Therefore, for any $x \in \Omega_1 \cap \Omega_2$, we get

$$\langle v_1+v_2, x-\bar{x}\rangle + \nu(x,\bar{x}) \le 0$$

which implies that $v_1+v_2 \in N^\nu(\bar{x},\Omega_1\cap\Omega_2)$. Thus, we get

$$N^\nu(\bar{x},\Omega_1\cap\Omega_2) \supset N(\bar{x},\Omega_1) + N^\nu(\bar{x},\Omega_2).$$

Hence, the proof is completed.

3. Subdifferential with degree of freedom of functions

This section presents a new version of subdifferential for functions, namely subdifferential with degree of freedom of functions. This is a generalization of the convex subdifferential known in convex analysis.

Definition 5. Let ν be a like-distance function X and let $f: X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$. The subdifferential with degree of freedom (or ν-subdifferential) of f at $\bar{x} \in \text{dom } f$ is defined by
$\partial^\nu f(\bar{x}) := \{x^* \in X^* \mid f(x) \ge f(\bar{x}) + \langle x^*, x-\bar{x}\rangle + \nu(x,\bar{x}) \forall x \in \text{dom } f\}$.

Theorem 6. Let $f: X \to \overline{\mathbb{R}}$ and $\nu: X\times X \to \mathbb{R}$ be a like-distance function. Then we get

$$\partial^\nu(\lambda f)(x) = \lambda\, \partial^{\frac{\nu}{\lambda}} f(x)\ \forall \lambda > 0, x \in \text{dom } f$$

Proof. Let $x \in \text{dom } f, \lambda > 0$ and $x^* \in \partial^\nu(\lambda f)(x)$. We have by the definition that

$$\lambda f(u) \ge \lambda f(x) + \langle x^*, u-x\rangle + \nu(u,x) \forall u \in \text{dom } f$$

which is equivalent to

$$f(u) \ge f(x) + \left\langle \frac{x^*}{\lambda}, u-x\right\rangle + \frac{1}{\lambda}\nu(u,x) \forall u \in \text{dom } f.$$

This means that $0 \in \partial^{\frac{1}{x}\nu} f(x)$
Hence, the proof of theorem is completed.

We next provide computation rules for subdifferential with degree of freedom of functions as follows.

Theorem 7. Let $f_1, f_2: X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ be convex functions on X and let $\nu : X\times X \to \mathbb{R}$ be a like-distance function which is concave and non-negative on dom f_1. Assume that the following qualification condition is satisfied at $\bar{x} \in \text{dom } f_1 \cap \text{dom } f_2$:

$$[-\partial^\infty f_1(\bar{x})] \cap \partial^\infty f_2(\bar{x}) = \{0\}$$

Then, we get

$$\partial^\nu(f_1+f_2)(\bar{x}) = \partial^\nu f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Proof. Let $v \in \partial^\nu(f_1+f_2)(\bar{x})$. By the definition, we get

$$f_1(x)+f_2(x) \ge f_1(\bar{x}) + f_2(\bar{x}) + \langle v, x-\bar{x}\rangle + \nu(x,\bar{x}), \forall x \in X. \quad (1.9)$$

We define the following sets

$$\Omega_1 := \{(x,\lambda_1,\lambda_2) \in X\times\mathbb{R}\times\mathbb{R} \mid \lambda_1 \ge f_1(x)\}$$

and

$$\Omega_2 := \{(x,\lambda_1,\lambda_2) \in X\times\mathbb{R}\times\mathbb{R} \mid \lambda_2 \ge f_2(x)\}.$$

Then by the convexity of f_1 and f_2, Ω_1 and Ω_2 are also convex. We define $\tilde{\nu}: (X\times\mathbb{R}\times\mathbb{R}) \times (X\times\mathbb{R}\times\mathbb{R}) \to \mathbb{R}$ by $\tilde{\nu}\big((x_1,\lambda_1,\beta_1),(x_2,\lambda_2,\beta_2)\big) := \nu(x_1,x_2)$ as a concave function due to the concavity of ν. We will show that $(v,-1,-1) \in N^{\tilde{\nu}}\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_1\cap\Omega_2\big)$.
Indeed, we get from (1.1) that $\langle v, x-\bar{x}\rangle - 1(\lambda - f_1(\bar{x})) - 1(\beta - f_2(\bar{x})) + \tilde{\nu}\big((x,\lambda,\beta),(\bar{x},f_1(\bar{x}),f_2(\bar{x}))\big) \le 0, \forall (x,\lambda,\beta) \in \Omega_1\cap\Omega_2$ meaning that $(v,-1,-1) \in N^{\tilde{\nu}}\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_1\cap\Omega_2\big)$. Moreover, by Corollary 2, we get

$$N^{\tilde{\nu}}\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_1\big) = N^{\bar{\nu}}\big((\bar{x},f_1(\bar{x})),\text{epi } f_1\big)\times\{0\}$$

where $\bar{\nu}: X\times\mathbb{R}\times X\times\mathbb{R} \to \mathbb{R}$ is define by $\bar{\nu}((x,\lambda),(y,\beta)) := \nu(x,y)$. Furthermore, it is obvious from the definition that

$$N\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_2\big) = \big\{(x^*,0,\beta) \in X^*\times\mathbb{R}\times\mathbb{R} \mid (x^*,\beta) \in N\big((\bar{x},f_2(\bar{x})),\Omega_2\big)\big\}.$$

To proceed further, it is necessary to prove that

$$\big[-N\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_1\big)\big] \cap N\Big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_2\Big) = \{(0,0,0)\}.$$

Indeed, let in the opposite that

$$(0,0,0) \ne (v_1,v_2,v_3) \in \big[-N\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_1\big)\big] \cap N\big((\bar{x},f_1(\bar{x}),f_2(\bar{x})),\Omega_2\big)$$

then $v_2 = v_3 = 0$ and $v_1 \ne 0$. By the definition, we get

$$\langle -v_1, x-\bar{x}\rangle \le 0\ \forall x \in \text{dom} f_1 \text{ and } \langle v_1, x-\bar{x}\rangle \le 0\ \forall x \in \text{dom } f_2$$

which imply that $v_1 \in (-\partial^\infty f_1(\bar{x})) \cap (\partial^\infty f_2(\bar{x}))$. This contradicts to the assumption that $[-\partial^\infty f_1(\bar{x})] \cap \partial^\infty f_2(\bar{x}) = \{0\}$. Therefore, by using Theorem 4, we find elements $(v_1,-1,0) \in$

$N^{\tilde{\nu}}\big((\bar{x}, f_1(\bar{x}), f_2(\bar{x})), \Omega_1\big)$ and $(v_2, 0, -1) \in N\big((\bar{x}, f_1(\bar{x}), f_2(\bar{x})), \Omega_2\big)$ such that

$$(v, -1, -1) = (v_1, -1, 0) + (v_2, 0, -1).$$

This follows that $v = v_1 + v_2$ with $v_1 \in \partial^\nu f_1(\bar{x})$ and $v_2 \in \partial f_2(\bar{x})$. Hence, the proof of the theorem is completed.

We next introduce a new version of the generalized convexity of functions as follows.

Definition 8. Let ν be a like-distance function on $C \subset X$ and let C be convex. The function $f: C \to \overline{\mathbb{R}}$ is called to be ν-convex at $\bar{x} \in \text{dom } f$ if dom f is a nonempty convex set and for any $x, y \in \text{dom } f$ and $\varphi(\cdot) := f(\cdot) - \nu(\cdot, \bar{x})$ is convex. f is called ν-convex on C if it is ν-convex at any $x \in C$.

Remark 9. f is 0 -convex on X if and only if f is convex.

It is known in convex analysis that the convex subdifferential of convex functions at any relative interior point of its domain are nonempty sets. We now present a similar result for ν-convex functions and ν-subdifferentials as follows.

Theorem 10. Let ν be a like-distance function X. Let $f: X \to \overline{\mathbb{R}}$ be a ν-convex function. Then $\partial^\nu f(x) \neq \emptyset$ for any $x \in$ re (dom f).

Proof. Fix $x \in$ re (dom f). Since $f(\cdot) - \nu(\cdot, x)$ is convex, there exists $x^* \in \partial(f(\cdot) - \nu(\cdot, x))(x)$ which means that $f(u) - \nu(u, x) \geq f(x) - \nu(x, x) + \langle x^*, u - x\rangle \forall x \in X$.

This is equivalent to

$$f(u) \geq f(x) + \langle x^*, u - x\rangle + \nu(u, x) \forall x \in X$$

which follows that $x^* \in \partial^\nu f(x)$.

4. Applications to optimization problems

The main goal of this section is to establish necessary and sufficient condition for a generalized solution to optimization problems. We first introduce generalized solutions to the optimization problems as follows.

Definition 11. Let ν be a like-distance function X. Let $f: X \to \overline{\mathbb{R}}$ and let $\bar{x} \in \text{dom } f$. Then $\bar{x}$ is called ν-minimizer of f if

$$f(x) \geq f(\bar{x}) + \nu(x, \bar{x}) \text{ for all } x \in \text{dom } f$$

Remark 12. (i) If $\nu(x, \bar{x}) \geq 0$ for all $x \in \text{dom } f$ then $\bar{x}$ is a global minimizer of f whenever $\bar{x}$ is ν-minimizer of f.

(ii) If $\nu(x, \bar{x}) = \ell \parallel x - \bar{x} \parallel$ with some $\ell > 0$, for all $x \in \text{dom } f$ then f is (global) calm at $\bar{x}$ if and only if $\bar{x}$ is ν-minimizer of f.

(iii) If $\nu(x, \bar{x}) = \ell \parallel x - \bar{x} \parallel^2$ with some $\ell > 0$, for all $x \in \text{dom } f$ then f is global second order growth at $\bar{x}$ if and only if $\bar{x}$ is ν-minimizer of f.

Theorem 13. Let ν be a like-distance function X. Let $f: X \to \overline{\mathbb{R}}$ be a ν-convex function. An element $\bar{x} \in \text{dom } f$ is a ν-minimizer of f if and only if

$$0 \in \partial^\nu f(\bar{x})$$

Proof. Let $\bar{x}$ be ν-minimizer of f. By the definition, we have

$$f(x) \geq f(\bar{x}) + \nu(x, \bar{x}) \text{ for all } x \in \text{dom } f$$

This is equivalent to

$$f(x) \geq f(\bar{x}) + \langle 0, x - \bar{x}\rangle + \nu(x, \bar{x}) \forall x \in \text{dom } f$$

This means that $0 \in \partial^\nu f(\bar{x})$.

Example 1. Consider the function $f(x) = e^x - 1 - x$ and $\nu(x, y) := \frac{1}{6}(x - y)^3$ for any $x, y \in \mathbb{R}$. It is clearly that

$$0 \in \partial^\nu f(0) = \{0\}.$$

Thus, $\bar{x} = 0$ is a ν-minimizer of f due to Theorem 13. We will directly check that $\bar{x}$ is a ν-minimizer of f. Indeed, the function $g(x) := e^x - 1 - x - \frac{1}{6}x^3$ is convex because of $g''(x) = e^x - x > 0$ for all $x \in \mathbb{R}$. Moreover, $\bar{x} = 0$ is the unique solution to the equation $g'(x) = 0$. This implies that $\bar{x} = 0$ is a global minimizer of g which means that $g(x) \geq g(\bar{x}) = 0$ for all $x \in \mathbb{R}$. Therefore,

$$f(x) = e^x - 1 - x \geq f(\bar{x}) + \nu(x, \bar{x}) \text{ for all } x \in \mathbb{R}$$

which follows that $\bar{x}$ is a ν-minimizer of f.

We close this section by establishing the necessary and sufficient optimality condition for the following problem:

$$\min f(x) \text{ such that } x \in C \qquad (1.1)$$

where C is a nonempty closed convex set and f is a convex function on X with $C \subset \text{dom } f$.

Let $\nu: X \times X \to \overline{\mathbb{R}}$ be a like-distance function on C. A point $\bar{x} \in C$ is called a ν-global solution to the problem (1.1) if

$$f(x) \geq f(\bar{x}) + \nu(x, \bar{x}) \forall x \in C.$$

Theorem 14. Consider the problem (1.1). Let ν be a like-distance function on C and f be ν-convex on C. Assume that $\bar{x} \in C$ and the qualification condition

$$[-\partial^\infty f(\bar{x})] \cap N(\bar{x}, C) = \{0\} \qquad (1.2)$$

holds. Then $\bar{x} \in C$ is the ν-global solution to the problem (1.1) if and only if

$$0 \in \partial^\nu f(\bar{x}) + N(\bar{x}, C).$$

Proof. Let $\bar{x}$ be a ν-global solution to the problem (1.1). It is equivalent to that $\bar{x}$ is a ν-minimizer of

$f + \delta_C$. According to Theorem 13, $\bar{x}$ is a ν-minimizer of $f + \delta_C$ if and only if $0 \in \partial^\nu(f + \delta_C)(\bar{x})$ which is equivalent to

$$0 \in \partial^\nu f(\bar{x}) + N(\bar{x}, C)$$

due to Theorem 7 and the qualification condition (1.2). Thus, $\bar{x}$ is ν-global solution to the problem (1.1) if and only if

$$0 \in \partial^\nu f(\bar{x}) + N(\bar{x}, C).$$

Hence, the proof is completed.

Remark 15. In the case of $\nu(\cdot, \bar{x}) = 0$ for all $x \in C$, Theorem 14 reduces to the exiting result in convex analyis (see [Rockafellar (1996)]).

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