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MATKOWSKI'S FIXED POINT THEOREM IN $\mathbb{R}^m$-b-METRIC SPACES

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Abstract

In this paper, we aim to extend the fixed point theorem in metric spaces to $\mathbb{R}^m$-b-metric spaces. By constructing iterated sequences and proving that they are Cauchy sequences, we have established and proven the Matkowski fixed point theorem in $\mathbb{R}^m$-b-metric spaces. In addition, an example is presented to illustrate the obtained result.

Keywords: Contraction, fixed point, $\mathbb{R}^m$-b-metric space.

ĐỊNH LÝ ĐIỂM BẤT ĐỘNG MATKOWSKI TRONG KHÔNG GIAN $\mathbb{R}^m$-b-METRIC

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Tóm tắt

Trong bài viết này, chúng tôi đặt mục tiêu mở rộng định lí điểm bất động trong không gian metric sang không gian $\mathbb{R}^m$-b-metric. Bằng phương pháp xây dựng dãy lặp và chứng minh dãy lặp là dãy Cauchy, chúng tôi đã thiết lập và chứng minh định lý điểm bất động Matkowski trong không gian $\mathbb{R}^m$-b-metric. Ngoài ra, chúng tôi đưa ra một ví dụ để chứng minh cho kết quả đạt được.

Từ khóa: Co, điểm bất động, không gian $\mathbb{R}^m$-b-metric.

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1. Introduction

In 1974, by replacing the range $\mathbb{R}_+$ in the notion of a metric space with $\mathbb{R}_+^m$, Perov introduced the notion of a $\mathbb{R}^m$-*metric space* (Perov, 1974). In 1993, Czerwik introduced the notion of a b-metric space. It is defined by adding the constant $s \geq 1$ in the right of the triangle inequality of the notion of a metric space. Before that, Coifman and Guzmán (1970) mentioned this notion with the name of *quasimetric space*. As a generalization of a b-metric space and a $\mathbb{R}^m$-metric space, Boriceanu (2009) introduced the notion of $\mathbb{R}^m$-*b-metric space*. Many authors have studied fixed point theorems in $\mathbb{R}^m$-b-metric spaces (Boriceanu, 2009).

The Banach contraction theorem, which is known as one of the basic theorems of analysis, was given by Banach (1922). Because of its wide applications, authors have still been studying and generalizing it in different directions. Many kinds of contraction maps have been introduced. Boy and Wong (1969) introduced a new contraction map in metric spaces by replacing the contraction constant $k \in [0,1)$ with the upper semi-continuous function from the right ψ on $\mathbb{R}_+$ satisfying $0 \leq \psi(t) < t$ for all $t > 0$. After that, Matkowski showed that when the condition of the upper semi-continuity of ψ is replaced by the condition of the increasing of ψ in the theorem of Boy and Wong, the result of this theorem still holds (Matkowski, 1975). For more details, we refer the reader to (Kannan, 1969; Ciric, 1974; Kirk & Shahzad, 2014).

First, we recall the following definitions to be used in the this paper.

Definition 1.1. Let $\mathbb{R}_+^m = \{x = (x_1, \dots, x_m): x_i \in \mathbb{R}_+, i = 1, \dots, m\}, e = (1,1,\dots,1) \in \mathbb{R}_+^m$. For all $x, y \in \mathbb{R}_+^m, x = (x_1, \dots, x_m), y = (y_1, \dots, y_m)$, we denote

(1) $x \preceq y$ if and only if $x_i \leq y_i$ for all $i = 1, \dots, m$.

(2) $x \prec y$ if and only if $x_i < y_i$ for all $i = 1, \dots, m$.

(3) The norm $\|\cdot\|$ in $\mathbb{R}_+^m$ is called *monotone with respect to the partial ordering* $\preceq$ *in* $\mathbb{R}_+^m$ if for all $x, y \in \mathbb{R}_+^m,\ x \preceq y$, then $\| x \| \leq \| y \|$.

Definition 1.2 (Perov, 1964)**.** Let X be a nonempty set and a function $d: X \times X \to \mathbb{R}_+^m$ satisfy for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \preceq d(x, z) + d(z, y)$.

Then d is called a $\mathbb{R}^m$-*metric* and (X, d, s) is called a $\mathbb{R}^m$-*metric space*.

Definition 1.3 (Czerwik, 1993, page 5)**.** Let X be a nonempty set, $s \geq 1$ and a function $d: X \times X \to \mathbb{R}_+$ satisfy for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq s(d(x, z) + d(z, y))$.

Then d is called a *b-metric* and (X, d, s) is called a *b-metric space*.

Definition 1.4 (Boriceanu, 2009, Definition 2.1)**.** Let X be a nonempty set, $s \geq 1$ and a function $d: X \times X \to \mathbb{R}_+^m$ satisfy for all $x, y, z \in X$,

(1) $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x,y) \preceq s(d(x,z) + d(z,y))$.

Then we have

(1) d is called a $\mathbb{R}^m$-*b-metric* and (X,d,s) is called a $\mathbb{R}^m$-b-metric space.

(2) The sequence $\{z_n\}$ is called *convergent to* z if $\lim_{n\to\infty} d(z_n,z) = (0,\dots,0)$, written by $\lim_{n\to\infty} z_n = z$.

(3) The sequence $\{z_n\}$ is called *Cauchy* if $\lim_{n,m\to\infty} d(z_n,z_m) = (0,\dots,0)$.

(4) The $\mathbb{R}^m$-b-metric space (X,d,s) is called *complete* if every Cauchy sequence is a convergent sequence.

Remark 1.5. (1) Some authors also call the $\mathbb{R}^m$-b-metric space as the *generalized b-metric space* (Bazine, 2022) or the *vector-valued b-metric space* (Boriceanu, 2009).

(2) If $m = 1$, then a $\mathbb{R}^m$-b-metric space (X,d,s) is a b-metric space in the sense of Czerwik (Czerwik, 1998).

(3) If $s = 1$, then a $\mathbb{R}^m$-b-metric space $(X,d,1)$ is a $\mathbb{R}^m$-metric space in the sense of Perov (Perov, 1964).

Theorem 1. 6 (Kirk & Sims, 2001, Theorem 3.4)**.** Assume that

(1) (X,d) is a metric space and a function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is such that

(a) ψ is increasing, that is, for all $x < y$ implies $\psi(x) \leq \psi(y)$.

(b) For all $t \in \mathbb{R}_+$, $\lim_{n\to\infty} \psi^n(t) = 0$.

(2) The map $f: X \to X$ satisfies for all $x,y \in X$,

$$d\big(fx, f(y)\big) \leq \psi(d(x,y)).$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n x = x^*$ for all $x \in X$.

In this paper, we establish Matkowski's fixed point theorem in $\mathbb{R}^m$-b-metric spaces. Moreover, we give an example to illustrate the obtained result.

2. Main results

First, we establish the Matkowski's fixed point theorem in $\mathbb{R}^m$-b-metric spaces as follows.

Theorem 2.1. Assume that

(1) (X,d,s) is a $\mathbb{R}^m$-b-metric space and a function $\psi: \mathbb{R}_+^m \to \mathbb{R}_+^m$ is such that
(a) ψ is increasing, that is, for all $x \prec y$ implies $\psi(x) \preceq \psi(y)$.
(b) For all $t \in \mathbb{R}_+^m$,

$$\lim_{n\to\infty} \psi^n(t) = 0. \tag{2.1}$$

(2) The map $f: X \to X$ satisfies for all $x,y \in X$,

$$d(fx, f(y)) \preceq \psi(d(x,y)). \tag{2.2}$$

Then f has a unique fixed point x^* and $\lim_{n\to\infty} f^n x = x^*$ for all $x \in X$.

Proof. From (2.1) we deduce that $\lim_{n\to\infty} \psi^n((1,\dots,1)) = (0,\dots,0)$. Then there exists n_0 such that

$$\psi^{n_0}((1,\dots,1)) \prec \left(\tfrac{1}{2s},\dots,\tfrac{1}{2s}\right). \tag{2.3}$$

Let $x \in X$. Put $g = f^{n_0}$ and put $x_m = g^m(x)$ for all $m \in \mathbb{N}$. By (2.2) we deduce

$$d(x_{m+1}, x_m) = d\left(g^m\big(g(x)\big), g^m(x)\right) \preceq \cdots \preceq \psi^{m.n_0}(d(g(x), x)). \tag{2.4}$$

Taking the limit as $m \to \infty$ in (2.4), we get $\lim_{m\to\infty} d(x_{m+1}, x_m) = (0,\dots,0)$. So there exists m_0 such that for all $m \geq m_0$,

$$d(x_{m+1}, x_m) \prec \left(\tfrac{1}{2s},\dots,\tfrac{1}{2s}\right). \tag{2.5}$$

We denote $B[x,r] = \{y \in \mathbb{R}^m : d(x,y) \preceq r\}$

where $x \in \mathbb{R}^m, r \in \mathbb{R}_+$. Now for each $u \in B[x_{m_0}, 1]$ and by (2.3) we have

$$\begin{aligned} d\left(g(u), g(x_{m_0})\right) &= d\left(f^{n_0}(u), f^{n_0}(x_{m_0})\right) \\ &\preceq \psi^{n_0}\left(d(u, x_{m_0})\right) \\ &\preceq \psi^{n_0}((1,\dots,1)) \\ &\prec \left(\tfrac{1}{2s},\dots,\tfrac{1}{2s}\right). \end{aligned} \tag{2.6}$$

We first show that g has a fixed point x^*. Indeed, let $u \in B[x_{m_0}, 1]$. By (2.5) and (2.6) we have

$$\begin{aligned} d\big(g(u), x_{m_0}\big) & \\ &\preceq s\left[d\left(g(u), g(x_{m_0})\right) + d\big(g(x_{m_0}), x_{m_0}\big)\right] \\ &\preceq s\left[\left(\frac{1}{2s},\dots,\frac{1}{2s}\right) + \left(\frac{1}{2s},\dots,\frac{1}{2s}\right)\right] \\ &= (1,\dots,1). \end{aligned}$$

So, $g(u) \in B[x_{m_0}, 1]$. Then, we conclude that $g: B[x_{m_0}, 1] \longrightarrow B[x_{m_0}, 1]$. For all $n, m \geq m_0$ we find that

$$\begin{aligned} d(x_n, x_m) &\preceq s\big[d(x_n, x_{m_0}) + d(x_{m_0}, x_m)\big] \\ &\preceq s[(1,\dots,1) + (1,\dots,1)] \\ &= (2s,\dots,2s). \end{aligned} \tag{2.7}$$

By (2.2) we find that for $m \geq n \geq m_0$,

$$
\begin{aligned}
&d(x_n, x_m) = d(g^n(x), g^m(x)) \\
&= d(g^{n-m_0} g^{m_0}(x), g^{m-m_0} g^{m_0}(x)) \\
&= d\left(g^{n-m_0}(x_{m_0}), g^{m-m_0}(x_{m_0})\right) \\
&= d\left(f^{(n-m_0)n_0}(x_{m_0}), f^{(m-m_0)n_0}(x_{m_0})\right) \\
&\preceq \ \psi\left(d\left(f^{(n-m_0)n_0-1}(x_{m_0}), f^{(m-m_0)n_0-1}(x_{m_0})\right)\right) \\
&\preceq \psi^{(n-m_0)n_0}\left(d\left(x_{m_0}, f^{(m-m_0)n_0-(n-m_0)n_0}(x_{m_0})\right)\right) \\
&= \psi^{(n-m_0)n_0}\left(d\left(x_{m_0}, f^{(m-n)n_0}(x_{m_0})\right)\right) \\
&= \psi^{(n-m_0)n_0}\left(d\left(x_{m_0}, x_{m-n+m_0}\right)\right) \\
&\preceq \psi^{(n-m_0)n_0}((2s, \dots, 2s)).
\end{aligned}
$$

Taking the limit as $n, m \to \infty$ in (2.7) and using (2.1) we find that $\lim_{n,m\to\infty} d(x_n, x_m) = (0, \dots, 0)$. So $\{x_m\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $\lim_{m\to\infty} x_m = x^*$.

From the assumption of ψ we find that $\lim_{t\to(0^+,\dots,0^+)} \psi(t) = (0, \dots, 0)$. Then for all $\{y_n\} \subset X$, $\lim_{n\to\infty} y_n = y$, we have

$$\lim_{n\to\infty} d(f(y_n), f(y)) \preceq \lim_{n\to\infty} \psi\big(d(y_n, y)\big) = (0, \dots, 0).$$

Then we get

$$\lim_{n\to\infty} f(y_n) = f(y). \tag{2.8}$$

We have that

$$\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} f^{n_0}(y_n) = f^{n_0}(y) = g(y). \tag{2.9}$$

Then

$$x^* = \lim_{m\to\infty} x_m = \lim_{m\to\infty} x_{m+1} = \lim_{m\to\infty} g(x_m) = g(x^*).$$

So g has a fixed point. From (2.2) we have

$$
\begin{aligned}
d(x^*, g^m(f(x))) \quad &= d(g^m(x^*), g^m(f(x))) \\
&= d(f^{n_0 m}(x^*), f^{n_0 m}(f(x))) \\
&\preceq \psi^{n_0 m}\big(d(x^*, f(x))\big)
\end{aligned} \tag{2.10}
$$

and

$$
\begin{aligned}
d(x^*, g^m(x)) \quad &= d(g^m(x^*), g^m(x)) \\
&= d(f^{n_0 m}(x^*), f^{n_0 m}(x)) \ \text{(2.11)} \\
&\preceq \psi^{n_0 m}\big(d(x^*, x)\big).
\end{aligned}
$$

Taking the limit as $m \to \infty$ in (2.10) and (2.11) we get

$$\lim_{m\to\infty} d(x^*, g^m(f(x))) = \lim_{m\to\infty} d(x^*, g^m(x)) = (0, \dots, 0).$$

Then $\lim_{m\to\infty} g^m(f(x)) = \lim_{m\to\infty} g^m(x) = x^*$ in (X, d). By (2.10) we have

$$f(x^*) = \lim_{m\to\infty} f(g^m(x)) = \lim_{m\to\infty} f(f^{n_a m}(x)) = \lim_{m\to\infty} g^m(f(x)) = x^*.$$

This proves that x^* is a fixed point of f.

Next, we prove the uniqueness of fixed points of f. On the contrary, let x^* and y^* be two distinct fixed points of f. Then $d(x^*, y^*) \succ (0, \dots, 0)$. Therefore,

$$d(x^*, y^*) = d\big(f(x^*), f(y^*)\big) \preceq \psi\big(d(x^*, y^*)\big) \prec d(x^*, y^*).$$

It is a contradiction.

Finally, we show that $\lim_{n\to\infty} f^n(x) = x^*$. Note that $\lim_{m\to\infty} g^m(y) = x^*$ for all $y \in X$. For each $n \in \mathbb{N}$, there exists l_n such that $n = l_n n_0 + r_n$ with $0 \le r_n \le n_0 - 1$. So

$$f^n(x) = f^{l_n n_0 + r_n}(x) = g^{l_n}(f^{r_n}(x)).$$

Fix $r_n = r \in [0, n_0 - 1]$. Then

$$\lim_{l_n\to\infty} f^{l_n n_0 + r}(x) = \lim_{l_n\to\infty} g^{l_n}(f^r(x)) = x^*.$$

It implies that $\lim_{n\to\infty} f^n(x) = x^*$.

From Theorem 2.1, we infer the following corollaries.

Corollary 2.2 (Kirk & Shazad, 2014, Theorem 12.2)**.** Assume that

(1) (X, d, s) is a b-metric space and a function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is such that

(a) ψ is increasing.

(b) For all $t \in \mathbb{R}_+$, $\lim_{n\to\infty} \psi^n(t) = 0$.

(2) The map $f: X \to X$ satisfies for all $x, y \in X$,

$$d(f(x), f(y)) \le \psi(d(x, y)).$$

Then f has a unique fixed point x^* and for all $x \in X$ $\lim_{n\to\infty} f^n(x) = x^*$.

Corollary 2.3 (Kirk & Sims, 2001, Theorem 3.4)**.** Assume that

(1) $(X, d,$ s) is a metric space and a function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ is such that

(a) ψ is increasing.

(b) For all $t \in \mathbb{R}_+$, $\lim_{n\to\infty} \psi^n(t) = 0$.

(2) The map $f: X \to X$ satisfies for all $x, y \in X$,

$$d(f(x), f(y)) \le \psi(d(x, y)).$$

Then f has a unique fixed point x^* and for all $x \in X$ $\lim_{n\to\infty} f^n(x) = x^*$.

Now, we give an example to illustrate the obtained result.

Example 2.4. Assume that

(1) $X = [0, \infty)$ and for all $x, y \in X$,

$d(x, y) = (|x - y|^2, |x - y|^2)$.

(2) A function $\psi: [0,\infty) \times [0,\infty) \to [0,\infty) \times [0,\infty)$ satisfies

$$\psi(x,y) = \left(\frac{x}{1+x}, \frac{y}{1+y}\right), \text{ for all } x, y \in [0,\infty).$$

(3) A map $f: [0,\infty) \to [0,\infty)$ satisfies

$$f(x) = \frac{x}{1+x}, \text{ for all } x \in [0,\infty).$$

Then f has a unique fixed point $x^* = (0,0)$.

Proof. We have $(X, d, 2)$ is a complete $\mathbb{R}^2$-b-metric space. Since $h'(t) = \frac{1}{(1+t)^2}$, where $h(t) = \frac{t}{1+t}$, $t \in [0,\infty)$, then h is increasing. It implies that ψ is increasing. Moreover, we have

$$\lim_{n\to\infty} \psi^n(x,y) = \lim_{n\to\infty} \left(\frac{x}{1+nx}, \frac{y}{1+ny}\right) = (0,0).$$

For all $x, y \in X$, we have

$$\begin{aligned} d(f(x), f(y)) &= \left(\left|\frac{x}{1+x} - \frac{y}{1+y}\right|^2, \left|\frac{x}{1+x} - \frac{y}{1+y}\right|^2\right) \\ &\leq \left(\left(\frac{|x-y|}{1+|x-y|}\right)^2, \left(\frac{|x-y|}{1+|x-y|}\right)^2\right) \\ &\leq \left(\frac{|x-y|^2}{1+|x-y|^2}, \frac{|x-y|^2}{1+|x-y|^2}\right) \\ &\leq \psi(d(x,y)). \end{aligned}$$

By Theorem 2.1, we have f has a unique fixed point x^* and $\lim_{n\to\infty} f^n(x) = x^*$ for all $x \in X$. For $x = (1,1)$, then $\lim_{n\to\infty} f^n(1,1) = \lim_{n\to\infty} \left(\frac{1}{n+1}, \frac{1}{n+1}\right) = (0,0)$. Therefore, $x^* = (0,0)$ is a unique fixed point of f.

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