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INTERVAL VALUED OPTIMIZATION PROBLEM ON HADAMARD MANIFOLD: WOLFE DUALITY

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Abstract

This paper will study about the duality for interval valued optimization problems on Hadamard manifolds. That is Wolfe dual problem with weak duality and strong duality. These results are the complement for the solvability of interval valued optimization problems on Hadamard manifolds.

Keyword: *Interval valued function, Hadamard manifold, gH-differentiable, Wolfe duality.*

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TỐI ƯU GIÁ TRỊ KHOẢNG TRÊN CÁC ĐA TẠP HADAMARD: ĐỐI NGẪU WOLFE

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Tóm tắt

Trong bài viết này, chúng tôi giới thiệu về đối ngẫu của các bài toán tối ưu giá trị khoảng trên các đa tạp Hadamard. Đó là đối ngẫu Wolfe cùng với các điều kiện đối ngẫu yếu và đối ngẫu mạnh. Các kết quả này sẽ hoàn thiện thêm các cách để tiếp cận bài toán tối ưu giá trị khoảng trên các đa tạp Hadamard.

Từ khoá: *Hàm giá trị khoảng, Đối ngẫu Wolfe, Đa tạp Hadamard,* gH *-khả vi.*

1. Introduction

There are a lot of problems, which may not be solved on Euclidean space and require more general space. Indeed, the extending of optimization problems was studied about forty years ago. Some types and algorithms of optimization problems have been extended from Euclidean space to Riemannian manifolds, specially to Hadamard manifolds (Absil et al., 2008; Bacak, 2014).

Nowadays, the uncertainty handling optimization techniques are most powerful to increase the productivity. In general, fuzzy, stochastic and grey optimization techniques are some approaches to tackle these problems. Each of them has some strengths and limitations. While formulating mathematical models from the available data, one may replace those by intervals. It could be customer's age, monthly electricity consumption or kiln temperature, etc., (see Diamond & Kloeden, 1994) for more information. The set of all intervals ($I(\mathbb{R})$) does not have linearly order, then almost usually methods to solve optimization problems might not be easily applied for interval optimizations problems (IOPs). Bhurjee and Panda (2013) gave a notion of efficient solutions of interval optimization problems, which is similar to the Pareto optimality concept in multi-objective optimization problems. Based on this idea, some authors studied the optimality condition for IOPs (Jana & Panda 2014). Gosh (2017) applied the Newton method and quasi-Newton method with rank-two to obtain efficient point of the IOPs, which exploit the parametric representation technique. Ishibuchi and Tanaka (1990) presented the IOPs with linear objective by using the multi-objective programming. However, all of them were studied on Euclidean space. To our best knowledge, there is a few studying on Riemannian interval optimization problems (RIOPs) in the literature.

Recently, Nguyen et al. (2023, 2024) introduced the new type of optimization problem, which is called interval valued optimization problem on Hadamard manifolds. It means the valued of objective functions are closed interval in $\mathbb{R}$. In those paper, the author studied about gH - convex, gH -differential of interval valued functions. They also studied about the solvability and KKT conditions of RIOPs. The steepest descent method and the partial convergence were obtained. To the constrained RIOPs, they studied the exact penalty approach to convert a constrained RIOP to be an unconstrained RIOP.

Wolfe (1961) presented the Wolfe duality for real valued optimization problem on Euclidean space. Wu (2008) study a concept for interval valued objective functions. In this paper, we will extend the space from $\mathbb{R}^n$ to Hadamard manifolds. In other way, we also use the concept of general Hukuhara difference, which is the global difference instead of the partial difference in Wu (2008).

Consider the optimization problem as

$$\begin{aligned} &\min f(x) \\ &s.t. \quad x \in D, G_i(x) \preceq [0,0], i = 1,\dots,r, \end{aligned}$$

where $f, G_i : \mathcal{M} \to I(\mathbb{R}), i = 1,\dots,r$ are given interval valued functions. The Lagrange function is defined as $L(x,\mu) = f(x) + \sum_{i=1}^{r} \mu_i G_i(x)$ and was studied by Nguyen et al. (2024). However, since the order on $I(\mathbb{R})$ is partial order, then the Lagrange dual problem does not exist. In this paper, we will study about Wolfe duality problem for RIOPs. It includes the construction of duality problem and the properties of duality gap. The interval valued optimization problems on Hadamard manifolds was studied by Nguyen et al. (2023, 2024). In these papers, for solvability, the authors presented some methods such as using the properties of the corresponding real-valued functions, the KKT conditions, the steepest descent method, and the exact penalty approach. We hope that our results will contribute the other ways for the solvability of RIOPs.

2. Preliminaries

For convenience, we will recall some basic knowledge about Riemannian manifold, specially Hadamard manifold, which can be found in some textbooks, such as Do Carmo (1992) or Jost (2011) and the reference therein. The interval, computing, the order on $I(\mathbb{R})$, Riemannian valued function (RIVF) and their properties are also important for next section.

Let us denote by $I(\mathbb{R})$ the set of all closed, bounded interval in $\mathbb{R}$, i.e.

$$I(\mathbb{R}) = \left\{[a^L, a^U], a^L, a^U \in \mathbb{R}, a^L \le a^U\right\}$$

Let $A = [a^L, a^U], B = [b^L, b^U] \in I(\mathbb{R}), k \in \mathbb{R}$, we have

$$A + B = [a^L + b^L, a^U + b^U]$$

$$kA = \begin{cases} [ka^L, ka^U] & if\ k \ge 0 \\ [ka^U, ka^L] & if\ k < 0 \end{cases}$$

There are some differences on $I(\mathbb{R})$ and this paper will study on general Hukuhara difference (gH -difference).

Definition 1 (gH -difference (Stefanini, 2008)). Let $A, B \in I(\mathbb{R})$. The gH -difference between A and B is defined as the interval C such that

$$C = A -_{gH} B \Leftrightarrow \begin{cases} A = B + C \\ or \\ B = A - C \end{cases}$$

Proposition 2 (Stefanini, 2008). For any two intervals $A = [a^L, a^U], B = [b^L, b^U]$, the gH -difference $C = A -_{gH} B$ always exists and

$$C = [\min\{a^L - b^L, a^U - b^U\}, \max\{a^L - b^L, a^U - b^U\}].$$

There does not exists the natural order on $I(\mathbb{R})$, hence we need to define it.

Definition 3 (Nguyen et al., 2023) Let $A = [a^L, a^U], B = [b^L, b^U] \in I(\mathbb{R})$, we write $A \preceq B$ if $a^L \le b^L$ and $a^U \le b^U$. We write $A \prec B$ if $A \preceq B$ and $A \ne B$. Equivalently, $A \prec B$ if and only if one of the following cases holds:

- $a^L < b^L$ and $a^U \le b^U$.
- $a^L \le b^L$ and $a^U < b^U$.
- $a^L < b^L$ and $a^U < b^U$.

We write $A \not\prec B$ if none of the above three cases hold. If neither $A \prec B$ nor $B \prec A$, we say that none of A and B dominates the other. Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of closed intervals. We write $\mathcal{A} \preceq \mathcal{B}$ if $A \preceq B$ for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

In this paper, when we call $\mathcal{M}$, it mean a Hadamard manifold, which is a simply connected, completed Riemannian manifold of nonpositive sectional curvature. For all $x, y \in \mathcal{M}$, by the Hopf-Rinow Theorem and Cartan-Hadamard Theorem (Jost, 2011), there exists a unique normalized geodesic joining x to y, which is a minimal geodesic. Let $D \subseteq \mathcal{M}$ be a nonempty set, a mapping $f : D \to I(\mathbb{R})$ is called a Riemannian interval valued function (RIVF). We write $f(x)=[f^L(x), f^U(x)]$ where f^L, f^U are real valued function satisfy $f^L(x) \le f^U(x), \forall x \in \mathcal{M}.$ Since $\mathbb{R} \subset I(\mathbb{R})$, then a Riemannian real valued function $f: D \to \mathbb{R}$ is also a Riemannian interval valued function.

Definition 4 (Nguyen et al., 2023). Let $D \subseteq \mathcal{M}$ be a geodesically convex set and $f: D \to I(\mathbb{R})$ be a RIVF. f is called geodesically convex on D if

$$f(\gamma(t)) \preceq (1-t)f(x)+tf(y), \quad \forall x, y \in D, \forall t \in [0,1],$$

where $\gamma:[0,1] \to \mathcal{M}$ is the minimal geodesic joining x and y. A RIVF f is called geodesically concave if $-f$ is geodesically convex.

Proposition 5 (Nguyen et al., 2023). Let D be a geodesically convex subset of $\mathcal{M}$ and f be a RIVF on D. Then, f is geodesiclly convex on D if and only if f^L and f^U are geodesically convex on D.

Definition 6 (gH -differentiability (Nguyen et al., 2023)). Let f be a RIVF on a nonempty open subset D of $\mathcal{M}$. The function f is said to have gH - directional derivative at $x \in D$ in direction $v \in T_x\mathcal{M}$, if there exists a closed bounded interval $f'(x,v)$ such that the limits

$$f'(x,v) = \lim_{t \to 0^+} \frac{1}{t}(f(exp_x(tv)) -_{gH} f(x))$$

exists, where $f'(x,v)$ is called the gH -directional derivative of f at x in the direction of v. If f has gH - directional derivative at x in every direction $v \in T_x\mathcal{M}$, we say that f is gH -directional differentiable at x.

Theorem 7 (Nguyen et al., 2023). Let $D \subseteq \mathcal{M}$ be a nonempty open geodesically convex set. If $f: D \to I(\mathbb{R})$ is a geodesically convex RIVF, then at any $x_0 \in D$, gH - directional derivative $f'(x,v)$ exists for every direction $v \in T_{x_0}\mathcal{M}.$

3. Wolfe duality for interval valued optimization problems on Hadamard manifolds

The Riemannian interval optimization problem (RIOP)

$$\begin{aligned} &\min f(x) \\ &s.t\ x \in \mathcal{M}, G_i(x) \preceq [0,0], i=1,\dots,r \end{aligned} \tag{1}$$

where $f, G_i: \mathcal{M} \to I(\mathbb{R}), i=1,\dots,r$ are RIVF. Problem (1) is called the primal problem, which was studied by Nguyen et al. (2023).

We denoted by

$$X = \{x \in \mathcal{M} : G_i(x) \preceq [0,0], i=1,\dots,r\}$$

The feasible set of RIOP (1). We also denote by

$$obj_P(f,X) := \{f(x) : x \in X\}$$

The set of all objective value of RIOP (1).

Definition 8 (Nguyen et al., 2023). Consider problem (1). A feasible point x_0 is said to be an efficient point of RIOP (1) if $f(x) \not\preceq f(x_0), \forall x \in X$. In this case, $f(x_0)$ is called efficient objective value of RIOP (1).

We denote by $\min(f, X)$ the set of all efficient objective of RIOP (1).

With assumption that $f, G_i, i = 1,...,r$ are geodesically convex, then they are gH-directional differentiable on $\mathcal{M}$. Let

$$H(x,\mu,v) = f'(x,v) + \sum_{i=1}^{n} \mu_i G_i'(x,v), \forall x \in \mathcal{M}, \mu_i \geq 0, i = 1,...,r, v \in T_x\mathcal{M}.$$

Note that, in general

$$L'(x,\mu,v) \neq H'(x,\mu,v).$$

Now, we will consider the Wolfe dual problem (WDRIOP) of primal problem (1) as follows:

$$\begin{aligned} &\max L(x,\mu) \\ &s.t\ x \in \mathcal{M}, H(x,\mu,v) = [0,0], \forall v \in T_x\mathcal{M} \\ &\mu \geq 0. \end{aligned} \tag{2}$$

Denote by

$$Y := \left\{(x,\mu) \in \mathcal{M} \times \mathbb{R}_+^r : H(x,\mu,v) = [0,0], \forall x \in \mathcal{M}, v \in T_x\mathcal{M}\right\}.$$

The set of all feasible points of WDRIOP, and denote by

$$ojb_D(L,Y) := \left\{L(x,\mu) : (x,\mu) \in Y\right\}.$$

the set of all objective values of Wolfe dual problem (2).

Definition 9. $(\bar{x},\bar{\mu}) \in Y$ is said to be an efficient point of WRIOP (2) if

$$L(\bar{x},\bar{\mu}) \not\preceq L(x,\mu), \forall (x,\mu) \in Y.$$

In this case, $L(\bar{x},\bar{\mu})$ is called efficient valued of WRIOP (2).

Example 1. Let $\mathcal{M} := \left\{x \in \mathbb{R} \mid x > 0\right\}$ be endowed with the Riemannian metric given by

$$\langle u,v\rangle_x = \frac{1}{x^2}uv, \forall u,v \in T_x\mathcal{M} = \mathbb{R}.$$

Then $\mathcal{M}$ is a Hadamard manifold. Consider the RIOP

$$\begin{aligned} &\min f(x) = \left[-x - \frac{1}{x}, x\right] \\ &x \in \mathcal{M}, G(x) \preceq [0,0]. \end{aligned}$$

Where $G(x) = [\min\{1 - \frac{1}{x-2}, -x\}, \max\{1 - \frac{1}{x-2}, -x\}]$. Hence, the feasible set

$$X = \{x \in \mathcal{M} : 1 - \frac{1}{x-2} \leq 0\} = \{x \in \mathbb{R} : 0 < x \leq 3\}.$$

For any $x \in \mathcal{M}$, we have

$$f'(x,v) = v[1, -1 + \frac{1}{(x-2)^2}], \forall v \in T_x\mathcal{M},$$

$$G'(x,v) = v[-1, \frac{1}{(x-2)^2}], \forall x \in T_x\mathcal{M}.$$

Therefore, we have the corresponding Wolfe dual problem

$$\begin{aligned} &\max L(x,\mu) \\ &s.t\ x \in \mathcal{M}, H(x,\mu,v) = [0,0], \forall v \in T_x\mathcal{M} \\ &\mu \geq 0. \end{aligned}$$

Where $L(x,\mu) = f(x) + \mu G(x),$

and $H(x,\mu,v) = f'(x,v) + \mu G'(x,v) = v[1-\mu, -1 + \frac{1}{(x-2)^2}(1+\mu)], \forall v \in T_x\mathcal{M}.$

The set of all feasible points of dual problem

$$\begin{aligned} Y &:= \left\{(x,\mu) \in \mathcal{M} \times \mathbb{R}^r_+ : H(x,\mu,v) = [0,0], v \in T_x\mathcal{M}\right\} \\ &= \left\{(x,\mu) \in \mathcal{M} \times \mathbb{R}^r_+ : 1-\mu = 0, -1 + \frac{1}{(x-2)^2}(1+\mu) = 0\right\} \\ &= \left\{(x,\mu) \in \mathcal{M} \times \mathbb{R}^r_+ : x = 2 \pm \sqrt{2}, \mu = 1\right\}. \end{aligned}$$

Theorem 10 [Duality]. Assume $x_0, (x_1, \mu)$ be the feasible points of RIOP (1) and WRIOP (2), respectively. If $f, G_i, i = 1,...,r$ are geodesically convex on $\mathcal{M}$ then $L(x_1,\mu) \preceq f(x_0)$.

Proof. Let $A_i = [a_i^L, a_i^U] \in I(\mathbb{R}), i = 1,...,n$, we will proof that if $\sum_{i=1}^n A_i \in \mathbb{R}$ then $A_i \in \mathbb{R}, i = 1,...,n.$ Indeed

$$\sum_{i=1}^n A_i \in \mathbb{R} \Rightarrow \sum_{i=1}^n a_i^L = \sum_{i=1}^n a_i^U. \tag{3}$$

If there exists $i \in \{1, ..., n\}$ such that $a_i^L < a_i^U$, then $\sum_{i=1}^n a_i^L < \sum_{i=1}^n a_i^U$. Which is contradict with (3). Hence, $a_i^L = a_i^U, i = 1,...,n$.

Since (x_1, μ) is a feasible point of DRIOP (2) then for some $v \in T_x\mathcal{M}$ we have

$$\begin{aligned} &H(x_1,\mu,v) = [0,0] \\ &\Rightarrow f'(x_1,v) + \sum_{i=1}^n \mu_i G_i'(x_1,v) = [0,0] \\ &\Rightarrow (f^L)'(x_1,v) + \sum_{i=1}^n \mu_i (G_i^L)'(x_1,v) = (f^U)'(x_1,v) + \sum_{i=1}^n \mu_i (G_i^U)'(x_1,v) = 0. \qquad (4) \end{aligned}$$

And f is geodesically convex RIVF on $\mathcal{M}$ then by Proposition 5, f^L, f^U are geodesically convex on $\mathcal{M}$, we have

$$f^L(x_0) \geq f^L(x_1) + (f^L)'(x_1, v). \tag{5}$$

Since G_i is geodesically convex and $\mu_i \geq 0, i = 1,...,r$ then

$$-\sum_{i=1}^{r} \mu_i (G_i^L)'(x_1, v) \geq \sum_{i=1}^{r} \mu_i [(G_i^L)(x_1) - (G_i^L)(x_0)] \geq \sum_{i=1}^{r} \mu_i (G_i^L)(x_1). \tag{6}$$

Hence, from (4), (5) and (6), we obtained

$$f^L(x_0) \geq f^L(x_1) + \sum_{i=1}^{r} \mu_i (G_i^L)'(x_1).$$

Similar, we also have

$$f^U(x_0) \geq f^U(x_1) + \sum_{i=1}^{r} \mu_i (G_i^U)'(x_1).$$

Hence, $L(x_1, \mu) \preceq f(x_0)$.

Corollary 11[Solvability 1]. Let $f, G_i, i = 1,...,r$ be geodesically convex on $\mathcal{M}$. Assume that x^* is a feasible point of primal problem (1) such that $f(x^*) \in obj_D(L, Y)$. Then x^* is an efficient point of primal problem (1).

Proof. Since $f(x^*) \in obj_D(L, Y)$, then there exists $(\bar{x}, \bar{\mu})$ is an feasible point of dual problem (2) such that

$$f(x^*) = L(\bar{x}, \bar{\mu}).$$

In other hand, by Theorem 9, we have

$$L(\bar{x}, \bar{\mu}) \preceq f(x), \forall x \in X.$$

Then x^* is an efficient point of primal problem (1).

Corollary 12 [solvability 2]. Let $f, G_i, i = 1,...,r$ be geodesically convex on $\mathcal{M}$. Assume that $(\bar{x}, \bar{\mu})$ is a feasible point of dual problem (2) satisfies $L(\bar{x}, \bar{\mu}) \in obj_P(f, X)$. Then $(\bar{x}, \bar{\mu})$ is an efficient point of dual problem (2).

Proof. Suppose that $(\bar{x}, \bar{\mu})$ is not an efficient point of dual problem (2), then there exists $(x, \mu) \in Y$ such that $L(\bar{x}, \bar{\mu}) \prec L(x, \mu)$. Since $L(\bar{x}, \bar{\mu}) \in obj_P(f, X)$ then there exists $x^* \in X$ such that $L(\bar{x}, \bar{\mu}) = f(x^*)$. It means $f(x^*) \prec L(x, \mu)$, which contradict with Theorem 9. Therefore, $(\bar{x}, \bar{\mu})$ is an efficient point of dual problem (2).

Corollary 13 [Weak duality]. Consider the primal problem (1) and the dual problem (2). Assume that $f, G_i, i = 1,...,r$ are geodesically convex on $\mathcal{M}$. Then, we have

$$A \preceq B, \forall A \in \max(L, Y), B \in \min(f, X).$$

Proof. Since $A \in \max(L, Y)$ then there exists $(x_1, \mu) \in Y$ such that $L(x_1, \mu) = A$. Similar, there exists $x_2 \in X$ such that $f(x_2) = B$. By Theorem 9, we have $L(x_1, \mu) \preceq f(x_2)$ or $A \preceq B$.

Definition 14 [Duality gap].

1.We say that the primal problem (1) and the dual problem (2) have no duality gap in the weak sense if $\min(f,X)\cap\max(L,Y)\neq\varnothing$.

2. We say that the primal problem (1) and the dual problem (2) have no duality gap in the strong sense if there exists $x^*\in\mathcal{M},\mu^*\geq 0$ such that

$$f(x^*)\in\min(f,X), L(x^*,\mu^*)\in\max(L,Y), \text{ and } f(x^*)=L(x^*,\mu^*).$$

Theorem 15 [Strong duality 1]. Let $f, G_i, i=1,\ldots,r$ be geodesically convex and gH-differentiable on $\mathcal{M}$. If one of the following conditions is satisfied

1, There exists a feasible point x^* of the primal problem (1) such that $f(x^*)\in obj_D(L,Y)$.

2. There exists a feasible point $(\bar{x},\bar{\mu})$ of the dual problem (2) such that $L(\bar{x},\bar{\mu})\in obj_D(f,X)$.

Then the primal problem (1) and the dual problem (2) have no gap in the weak sense.

Proof. The result follows immediately from Corollary 10 and Corollary 11.

Theorem 16 [Strong duality 2]. Let $f, G_i, i=1,\ldots,r$ be geodesically convex RIVF on $\mathcal{M}$. Suppose that, there exist $x^*\in\mathcal{M},\mu^*\geq 0$ such that $x^*,(x^*,\mu^*)$ are feasible points of the primal problem (1) and the dual problem (2), respectively, and

$$\sum_{i=1}^{r}\mu_i^* G_i(x^*)=[0,0]$$

Then the primal problem (1) and the dual problem (2) have no gap in the strong sense.

Proof. Assume that for any $x^*,(\bar{x},\bar{\mu})$ be the efficient points of primal problem (1) and the dual problem (2), respectively. We will prove that if $f(x^*)=L(\bar{x},\bar{\mu})$ then $x^*\in\min(f,X)$ and $(\bar{x},\bar{\mu})\in\max(L,Y)$. By Theorem 9, since $f(x^*)\in obj_D(L,Y)$ then $f(x^*)\in\min(f,X)$.

Similar, $L(\bar{x},\bar{\mu})\in obj_P(f,X)$ then by Theorem 9, $L(\bar{x},\bar{\mu})\in\max(L,Y)$.

Since $\sum_{i=1}^{r}\mu_i^* G_i(x^*)=[0,0]$, we have $f(x^*)=L(x^*,\mu^*)$ or the primal problem (1) and the dual problem (2) have no gap in the strong sense.

4. Conclusion

This work is the continuation of Nguyen's et al. (2023, 2024). It is also the generation of Wu's (Wu, 2008), not only extends the space from Euclidean space to Riemannian manifolds, but also uses the generalized difference, in fact general Hukuhara difference.

The Wolfe duality give us another way to study the constrained interval valued optimization problems. The duality theorems are the main results, including duality in weak and strong sence. We only work on Hadamard manifold, a special case of Riemannian manifold. In th future, we hope that the results can be generated on the general Riemannian manifold, which will help us to understand the optimizations problem on manifold. Otherwise, application of the results for some special manifold, such as Hyperpolic spaces or manifold of symmetric positive definite matrices, etc. is also very important.

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