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A COMPUTER ALGEBRA APPROACH FOR VERIFYING ISOMORPHISM OF LIE ALGEBRAS

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Abstract

This paper proposes a computer algebra method for verifying the isomorphism of finite-dimensional Lie algebras over the complex field. Afterwards, an example is analyzed to demonstrate the suggested method. Finally, its efficiency is shown through applications.

Keywords: *Isomorphism verifying, Lie algebras, Maple.*

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Tóm tắt

Trong bài báo này, chúng tôi đề xuất một phương pháp đại số máy tính để kiểm tra đẳng cấu giữa các đại số Lie hữu hạn chiều trên trường phức. Sau đó, một ví dụ được phân tích cụ thể để minh họa thuật toán. Cuối cùng, hiệu quả của phương pháp được trình bày thông qua một số ứng dụng.

Từ khóa: *Đại số Lie, Kiểm tra đẳng cấu, Maple.*

1. Introduction

One of the stages of classifying Lie algebras is to verify isomorphism of classified Lie algebras. For instance, the problem of classifying n-dimensional Lie algebras proceeds in two stages: (1) construct a list L consisting of n-dimensional Lie algebras, and (2) test isomorphism of Lie algebras in L. Thus, testing isomorphism of Lie algebras is a significant challenge.

Theoretically, two isomorphic Lie algebras have the same invariants (e.g., the dimension of the center, ideals in the derived/lower central series). However, the converse is not true in general. For two Lie algebras sharing the same invariants, it is generally impossible to determine whether they are isomorphic. Therefore, a natural way is to apply computer algebra to test isomorphism of Lie algebras.

In the relevant literature, the first authors considering the problem of testing isomorphism of Lie algebras from a computer algebra viewpoint are Gerdt and Lassner (1993). The authors translate the isomorphism conditions of Lie algebras into a system of polynomial equations. Groebner basis is then used to solve the latter problem. However, since the complexity of computing Groebner bases is very costly, this method is impractical when the dimensions of the input Lie algebras exceed 6. Moreover, a disadvantage of Groebner basis technique is that it is almost inapplicable for parametric Lie algebras.

It is known that the significant advancements in computer science has provided many effective tools for mathematics. In particular, one interesting tool for solving a polynomial system is the so-called triangular decomposition. Roughly speaking, triangular decomposition is a way of solving systems of polynomial equations that resembles the well-known Gaussian Elimination in Linear Algebra. Following the method of Gerdt and Lassner (1993), we also rewrite the isomorphic conditions of Lie algebras into polynomial systems. The triangular decomposition technique is then used instead of Groebner bases to solve these systems.

This paper is structured into five sections. This introduction presents the problem. Section 2 introduces the theory of triangular decomposition of polynomial systems. Then, Section 3 presents the main result of the paper with applications in Section 4. The final section contains some concluding remarks.

2. Triangular decomposition of polynomial systems

This section briefly recalls the so-called triangular decomposition. For more details of the theory of triangular decomposition, we refer to Chen (2011), Chen and Maza (2012). In this section, $\mathbf{k}$ is a field with algebraic closure $\mathbf{K}$. The notation $\mathbf{k}[\mathbf{x}]$ denotes the polynomial ring $\mathbf{k}[x_1,\ldots,x_n]$ with ordered variables $\mathbf{x}=\{x_1,\ldots, x_n\}$ and $x_1<\cdots<x_n$. Here, the notation $x_i<x_j$ means that x_i is less than x_j, as in the case where a is less than b in the lexicographic order.

Let $p\in\mathbf{k}[\mathbf{x}]\setminus\mathbf{k}$, i.e. p is a non-constant polynomial. The greatest variable of p is called the *main variable* of p and denoted by $\operatorname{mvar}(p)$. If $\operatorname{mvar}(p)=x_i$, we can consider p as a univariate polynomial in x_i, i.e., $p\in\mathbf{k}[x_1,\ldots,x_{i-1}][x_i]$. Then, the leading coefficient and leading monomial of p are called the *initial* and the *rank* of p which are denoted by $\operatorname{init}(p)$ and $\operatorname{rank}(p)$, respectively.

For $F \subset \mathbf{k}[\mathbf{x}]$, we denote by $\langle F \rangle$ the ideal in $\mathbf{k}[\mathbf{x}]$ spanned by F, and by $V(F)$ the zero set (solution set) of F in $\mathbf{K}^n$.

Let $I \subset \mathbf{k}[\mathbf{x}]$ be an ideal. A polynomial $p \in \mathbf{k}[\mathbf{x}]$ is called a *zerodivisor modulo I* if there exists $q \in \mathbf{k}[\mathbf{x}]$ such that $p \in I$ and $q \in I$ but $pq \in I$. If p is neither zero nor zerodivisor modulo I then p is called *regular modulo I*. For $h \in \mathbf{k}[\mathbf{x}]$, the *saturated ideal of I with respect to h* is an ideal in $\mathbf{k}[\mathbf{x}]$ as follows: $I : h^{\infty} := \{q \in \mathrm{k}[\mathrm{x}] : \exists m \in \mathbb{N} \text{ such that } h^m q \in I\}$.

A subset $T \subset \mathbf{k}[\mathbf{x}] \setminus \mathbf{k}$ consisting of polynomials with pairwise distinct main variables is called a *triangular set*. For a triangular set $T \subset \mathbf{k}[\mathbf{x}]$, the *saturated ideal* of T, denoted by $\operatorname{sat}(T) \subset \mathbf{k}[\mathbf{x}]$, is defined as follows: if $T = \varnothing$ then $\operatorname{sat}(T) = \{0\}$ is the trivial ideal. Otherwise, it is the ideal $\langle T \rangle : h_T^{\infty}$ where h_T is the product of the initials of the polynomials in T. The *quasi-component* of T is $W(T) := V(T) \setminus V(h_T)$.

Now, we turn to the main idea of triangular decomposition of polynomial systems. The most important object is the *regular chain* since it is the output of an algorithm to compute a triangular decomposition of polynomial systems.

Definition 2.1 (Regular chain)**.** A triangular set $T \subset \mathbf{k}[\mathbf{x}]$ is called a *regular chain* if:

(1) $T = \varnothing$; or

(2) $T \setminus \{T_{\mathbf{max}}\}$, where $T_{\mathbf{max}}$ is the polynomial in T with maximum rank, is a regular chain and $\operatorname{init}(T_{\mathbf{max}})$ is regular with respect to $\operatorname{sat}(T \setminus \{T_{\max}\})$.

Definition 2.2 (Triangular decomposition)**.** Let $F \subset \mathbf{k}[\mathbf{x}]$ be a finite set. A finite subset $\{T_1, \ldots, T_e\}$ consisting of regular chains of $\mathbf{k}[\mathbf{x}]$ is called a *triangular decomposition of* $V(F)$ if $V(F) = \bigcup_{i=1}^{e} W(T_i)$. If $V(F) = \varnothing$, i.e., system F has no solutions in $\mathbf{K}^n$, then the triangular decomposition of $V(F)$ is $\varnothing$.

For a polynomial system $F \subset R := \mathbf{k}[\mathbf{x}]$, Chen and Maza (2012) presented the algorithm $\mathbf{Triangularize}(F)$ to compute a triangular decomposition of $V(F) \subset \mathbf{K}^n$. In particular, this algorithm was implemented in Maple using the command $\mathbf{Triangularize}(F, R)$ from the RegularChains library.

Example 2.3. Consider the following system:

$$\begin{cases} x^2 + y + z = 1 \\ x + y^2 + z = 1 \\ x + y + z^2 = 1. \end{cases} \qquad (2.1)$$

According to Cox et al. (2015), the Groebner basis with respect to lexicographic order of ideal $\langle x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1 \rangle \subset \mathbb{C}[x, y, z]$ reduces to the fact that

$$(2.1) \Leftrightarrow \begin{cases} x + y + z^2 - 1 = 0 \\ y^2 - y - z^2 + z = 0 \\ 2yz^2 + z^4 - z^2 = 0 \\ z^6 - 4z^4 + 4z^3 - z^2 = 0. \end{cases}$$

Now, we set $F := \{x^2 + y + z - 1, x + y^2 + z - 1, x + y + z^2 - 1\} \subset \mathbb{C}[x, y, z]$. Then, **Triangularize**(F) returns four regular chains as follows:

$$T_1 = \{x - z, y - z, z^2 + 2z - 1\},\ T_2 = \{x, y, z - 1\},\ T_3 = \{x, y - 1, z\},\ T_4 = \{x - 1, y, z\}.$$

Such four regular chains have $h_{T_i} = 1$, so $W(T_i) = V(T_i) \setminus V(h_{T_i}) = V(T_i)$. Since $V(F) = \bigcup_{i=1}^{4} W(T_i)$, the solution set of (2.1) is divided into the solution sets of

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ z^2 + 2z - 1 = 0, \end{cases} \quad \begin{cases} x = 0 \\ y = 0 \\ z - 1 = 0, \end{cases}$$

$$\begin{cases} x = 0 \\ y - 1 = 0 \\ z = 0, \end{cases} \quad \begin{cases} x - 1 = 0 \\ y = 0 \\ z = 0. \end{cases}$$

Remark 2.4. As we can see in Example 2.3, Groebner bases are not necessarily triangular sets. Consequently, using a Groebner basis to construct isomorphisms is much more difficult, and in general, may be impossible.

3. The main result

This paper presents two main contributions: a theoretical background for testing the isomorphism of complex Lie algebras and an algorithm to perform the test.

Let $L = \operatorname{span}\{x_1, \ldots, x_n\}$ and $L' = \operatorname{span}\{y_1, \ldots, y_n\}$ be n-dimensional complex Lie algebras with structure constants $a_{ij}^k \in \mathbb{C}$ and $b_{ij}^k \in \mathbb{C}$, respectively. Here, $\operatorname{span}\{x_1, \ldots, x_n\}$ denotes an n-dimensional vector space with basis $\{x_1, \ldots, x_n\}$. Thus, we have

$$[x_i, x_j] = a_{ij}^k x_k,\quad [y_i, y_j] = b_{ij}^k y_k \quad (1 \le i < j \le n).$$

Assume that $\phi : L \to L'$ is a linear morphism whose matrix with respect to the two bases $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ is as follows:

$$[\phi] = \begin{bmatrix} z_{11} & \cdots & z_{n1} \\ \vdots & \ddots & \vdots \\ z_{1n} & \cdots & z_{nn} \end{bmatrix} \in \mathrm{Mat}_n(\mathbb{C}).$$

Then we have

$$\phi\left(\left[x_i, x_j\right]_L\right) = \sum_{s=1}^{n} \left(\sum_{k,l=1}^{n} z_{ik} z_{jl} b_{kl}^{s} \right) \cdot y_s, \quad \left[\phi(x_i), \phi(x_j)\right]_{L'} = \sum_{s=1}^{n} \left(\sum_{k=1}^{n} z_{ks} a_{ij}^{k} \right) \cdot y_s.$$

The linear morphism ϕ is a Lie algebra isomorphism if and only if

$$\begin{cases} \phi\left(\left[x_i, x_j\right]_L\right) = \left[\phi(x_i), \phi(x_j)\right]_{L'}, & 1 \le i < j \le n, \\ \det[\phi] \ne 0, \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} \sum_{k=1}^{n} z_{ks} a_{ij}^{k} - \sum_{k,l=1}^{n} z_{ik} z_{jl} b_{kl}^{s} = 0, & 1 \le i < j \le n, 1 \le s \le n, \\ 1 - z \cdot \det[\phi] = 0. \end{cases}$$

Consider $z_{ij}, z \in \mathbb{C}$ as variables. Note that the above system contains at most $n\binom{n}{2} + 1$ polynomials of degree at most $n+1$. Denote by $\mathbb{C}[\mathbf{z}] := \mathbb{C}\left[z_{ij}, z\right]$ the ring of polynomials in $n\binom{n}{2} + 1$ variables with coefficients in $\mathbb{C}$. Let

$$F(\mathbf{z}) = \left\{ \sum_{k=1}^{n} z_{ks} a_{ij}^{k} - \sum_{k,l=1}^{n} z_{ik} z_{jl} b_{kl}^{s}, 1 - z \cdot \det[\phi] : 1 \le i < j \le n, 1 \le s \le n \right\} \qquad (3.1)$$

be a polynomial system in $\mathbb{C}[\mathbf{z}]$. With these notations, the main result of the paper can be formulated as the following theorem.

Theorem 3.1. $L \cong L'$ if and only if $\mathbf{Triangularize}(F(\mathbf{z}))$ is non-empty.

Proof. As seen above, two n-dimensional complex Lie algebras L and L' are isomorphic if and only if the polynomial system $F(\mathbf{z})$ admits a solution $\mathbf{z} \in \mathbb{C}^{n^2+1}$, i.e., $\varnothing \ne V(F(\mathbf{z})) \subset \mathbb{C}^{n^2+1}$. By Definition 2.2, this is equivalent to $\mathbf{Triangularize}(F(\mathbf{z}))$ being non-empty. □

Using Theorem 3.1, we can summarize the procedure to test isomorphism between two n-dimensional complex Lie algebras by an algorithm as follows:

- Step 1. Set $R := \mathbb{C}[\mathbf{z}] = \mathbb{C}[z, z_{11}, \ldots, z_{1n}, \ldots, z_{n1}, \ldots, z_{nn}]$.

- Step 2. $V :=$ **Triangularize**$(F(\mathbf{z}))$ //command **Triangularize**$(F(\mathbf{z}), R)$.

- Step 3. If $V \neq \varnothing$ then $L \cong L'$, otherwise $L \ncong L'$.

4. Experimentations and applications

4.1. An example

We now present an example with detailed computations to demonstrate how Theorem 3.1 can be applied.

Example 4.1. Consider two 4-dimensional complex Lie algebras as follows:

$$L = \operatorname{span}\{x_1, x_2, x_3, x_4 : [x_2, x_3] = x_1, [x_2, x_4] = x_2, [x_3, x_4] = -x_3\},$$
$$L' = \operatorname{span}\{y_1, y_2, y_3, y_4 : [y_2, y_3] = y_1, [y_2, y_4] = -y_3, [y_3, y_4] = y_2\}.$$

We claim that $L \cong L'$. In fact, assume that $\phi : L \to L'$ is a Lie algebra isomorphism with matrix

$$[\phi] = \begin{bmatrix} z_{11} & z_{21} & z_{31} & z_{41} \\ z_{12} & z_{22} & z_{32} & z_{42} \\ z_{13} & z_{23} & z_{33} & z_{43} \\ z_{14} & z_{24} & z_{34} & z_{44} \end{bmatrix} \in \operatorname{Mat}_4(\mathbb{C}).$$

By computations, the system $F(\mathbf{z}) \subset \mathbb{C}[\mathbf{z}] = \mathbb{C}[z, z_{11}, \ldots, z_{14}, \ldots, z_{41}, \ldots, z_{44}]$ in (3.1) consists of twenty-two polynomials as follows:

$$F(\mathbf{z}) = \left\{ \begin{array}{l} 1 - z \cdot \det[\phi], z_{14}, z_{24}, z_{34}, z_{12}z_{23} - z_{13}z_{22}, z_{12}z_{33} - z_{13}z_{32}, z_{12}z_{43} - z_{13}z_{42}, \\ z_{12}z_{24} - z_{14}z_{22}, z_{12}z_{34} - z_{14}z_{32}, z_{12}z_{44} - z_{14}z_{42}, z_{13}z_{24} - z_{14}z_{23}, z_{13}z_{34} - z_{14}z_{33}, \\ z_{13}z_{44} - z_{14}z_{43}, z_{22}z_{33} - z_{23}z_{32} - z_{11}, z_{23}z_{34} - z_{24}z_{33} - z_{12}, z_{22}z_{34} - z_{24}z_{32} - z_{13}, \\ z_{22}z_{43} - z_{23}z_{42} - z_{21}, z_{22}z_{44} - z_{24}z_{42} + z_{23}, z_{23}z_{44} - z_{24}z_{43} - z_{22}, \\ z_{32}z_{43} - z_{33}z_{42} - z_{31}, z_{32}z_{44} - z_{34}z_{42} - z_{33}, z_{33}z_{44} - z_{34}z_{43} + z_{32} \end{array} \right\}.$$

Then, the **Triangularize**$(F(\mathbf{z}))$ algorithm, which computes a triangular decomposition of $V(F(\mathbf{z}))$, returns a unique regular chain as follows:

$$T = \left\{ \begin{array}{l} 4z_{33}^2 z_{44} z_{23}^2 z + 1, z_{11} - 2z_{33}z_{44}z_{23}, z_{12}, z_{13}, z_{14}, z_{21} + (z_{42} - z_{43}z_{44}) z_{23}, \\ z_{22} - z_{23}z_{44}, z_{24}, z_{31} - (z_{43}z_{44} + z_{42}) z_{33}, z_{32} + z_{33}z_{41}, z_{34}, z_{44}^2 + 1 \end{array} \right\}.$$

Since **Triangularize**$(F(\mathbf{z}))$ is non-empty, it follows from Theorem 3.1 that $L \cong L'$. Furthermore, $h_T = 4z_{33}^2 z_{44} z_{23}^2$ implies

$$V(F(\mathbf{z})) = W(T) = V(T) \setminus V(h_T) = V(T) \setminus V(4z_{33}^2 z_{44} z_{23}^2).$$

Thus, the polynomial system $F(\mathbf{z})=0$ is equivalent to the following system:

$$\begin{cases} 4z_{33}^2 z_{44} z_{23}^2 z+1 & =0 \\ z_{11}-2z_{33}z_{44}z_{23} & =0 \\ z_{12} & =0 \\ z_{13} & =0 \\ z_{14} & =0 \\ z_{21}+\left(z_{42}-z_{43}z_{44}\right)z_{23} & =0 \\ z_{22}-z_{23}z_{44} & =0 \\ z_{24} & =0 \\ z_{31}-\left(z_{43}z_{44}+z_{42}\right)z_{33} & =0 \\ z_{32}+z_{33}z_{41} & =0 \\ z_{34} & =0 \\ z_{44}^2+1 & =0 \\ 4z_{33}^2 z_{44} z_{23}^2 & \neq 0. \end{cases}$$

Solving this system yields the following isomorphism:

$$[\phi]=\begin{bmatrix} 2i & 0 & 0 & 0 \\ 0 & i & -i & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix},$$

where i is the imaginary unit.

4.2. Applications

From historical point of view, only full classification of solvable Lie algebras up to dimension 6 has been presented in the literature by Snobl and Winternitz (2014). Here, “full” means that they are classifications up to isomorphism. The same problem is still open beyond dimension 6.

According to Mubarakzyanov (1966), the nilradical $N(L)$ of a Lie algebra L over a field of characteristic zero always satisfies $2\dim N(L)\geq \dim L.$ Therefore, $\dim N(L)\in\{4,5,6,7\}$ if $\dim L=7.$ Hindeleh and Thompson (2008), Parry (2007) and Gong (1998) respectively classified 7-dimensional solvable Lie algebras L with $\dim N(L)\in\{4,6,7\}.$ The case $\dim N(L)=5$ remains open. Our contribution is the classification up to isomorphism all 7-dimensional complex solvable Lie algebras with 5-dimensional nilradicals (Le et al., 2023). This result completes the classification of complex solvable Lie algebras up to dimension 7.

5. Conclusion

This paper has presented an effective algorithm for testing isomorphism of finite-dimensional complex solvable Lie algebras. This algorithm has been applied to solve a fundamental problem in the theory of Lie algebras. In a forthcoming paper, an algorithm will be introduced for the more complicated case: testing isomorphism of parametric Lie algebras.

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