DONG THAP UNIVERSITY JOURNAL OF SCIENCE
Tạp chí Khoa học Đại học Đồng Tháp
Natural Sciences Issue
ISSN 0866-7675 | e-ISSN 2815-567X

**DOI: https://doi.org/10.52714/dthu.ns.2305.1834**

# CLASSIFYING 8-DIMENSIONAL SOLVABLE LIE ALGEBRAS HAVING 6-DIMENSIONAL ABELIAN NILRADICALS

**Le Minh Kha[1], Pham Quoc Thai[1], and Nguyen Thi Mong Tuyen[2*]**

[1]*Student, Dong Thap University, Cao Lanh 870000, Vietnam*

[2]*Faculty of Mathematics- Information Teacher Education, School of Education, Dong Thap University, Cao Lanh 870000, Vietnam*

**Corresponding author, Email: ntmtuyen@dthu.edu.vn*

*Article history*

*Received: 29/4/2025;* Received *in revised form: 28/7/2025; Accepted: 15/8/2025*

**Abstract**

*In this paper, we classify complex and real 8-dimensional solvable Lie algebras having a 6-dimensional abelian nilradical. The method is based on the fact that a given solvable Lie algebra L can be considered as an extension of its nilradical N(L), that is, the maximal nilpotent ideal of L. Therefore, we begin from a 6-dimensional abelian nilradical Lie algebra and, subsequently, we construct and classify all 8-dimensional solvable Lie algebras that admit it as their nilradical.*

**Keywords:** *Abelian nilradical, Lie algebra, Nilradical Lie algebra, Solvable Lie algebra.*

Cite: Nguyen, T. M. T., Le, M. K., & Pham, Q. T. (2026). Classification of 8-dimensional solvable Lie algebras having 6-dimensional Abelian nilradicals. *Dong Thap University Journal of Science, 15*(5), 60-72. https://doi.org/10.52714/dthu.ns.2305.1834

# PHÂN LOẠI ĐẠI SỐ LIE GIẢI ĐƯỢC 8-CHIỀU CÓ CĂN LŨY LINH GIAO HOÁN 6-CHIỀU

**Lê Minh Kha[1], Phạm Quốc Thái[1] và Nguyễn Thị Mộng Tuyền[2*]**

[1]*Sinh Viên, trường Đại học Đồng Tháp, Việt Nam*

[2]*Khoa Sư phạm Toán – Tin, Trường Sư phạm, Trường Đại học Đồng Tháp, Việt Nam*

[*]*Tác giả liên hệ, Email: ntmtuyen@dthu.edu.vn*

*Lịch sử bài báo*

*Ngày nhận: 29/4/2025; Ngày nhận chỉnh sửa: 28/7/2025; Ngày duyệt đăng: 15/8/2025*

**Tóm tắt**

*Trong bài báo này, chúng tôi phân loại lớp đại số Lie phức và thực giải được 8-chiều có căn lũy linh giao hoán 6-chiều. Phương pháp phân loại của chúng tôi là xem đại số Lie giải được L đã biết như là mở rộng của căn lũy linh N(L) của L, tức là ideal lũy linh cực đại của L. Do đó, chúng tôi bắt đầu từ một đại số Lie lũy linh giao hoán 6-chiều, sau đó xây dựng và phân loại tất cả các đại số Lie giải được 8-chiều nhận nó làm căn lũy linh.*

**Từ khóa:** *Căn lũy linh giao hoán, Đại số Lie, Đại số Lie lũy linh, Đại số Lie giải được.*

## 1. Introduction

The Levi-Malcev theorem (Levi, 1905; Malcev, 1945) reduces the classification of Lie algebras over a field of characteristic zero to that of semisimple and solvable Lie algebras. Semisimple Lie algebras have been fully classified over the complex field (Cartan, 1894) and the real field (Gantmacher, 1939), while that of solvable Lie algebras remains difficult and largely open. So far, a full classification of solvable Lie algebras of dimension 7 has been completed (Lie & Engel, 1893; Bianchi, 1903; Kruchkovich, 1954; Mubarakzyanov, 1963; Turkowski, 1990; Le et al., 2023), but the complete classification in dimension 8 is still unfinished.

Ndogmo and Winternitz (1994) established two fundamental theorems that form the foundation for the classification of solvable Lie algebras with abelian nilradicals. As a concrete application of their framework, they provided a complete classification of 7-dimensional solvable Lie algebras with 5-dimensional abelian nilradicals. This paper extends their approach to classify complex and real 8-dimensional solvable Lie algebras whose nilradicals are 6-dimensional abelian Lie algebras. This constitutes a special case within the broader classification scheme developed by Ndogmo and Winternitz.

This method is based on the fact that a given solvable Lie algebra $L$ can be considered as an extension of its nilradical $N(L)$, that is, the maximal nilpotent ideal of $L$. Therefore, we start from a 6-dimensional abelian nilpotent Lie algebra and classify all 8-dimensional solvable Lie algebras admitting it as their nilradicals.

## 2. Construction of Lie algebras

From now on, we will use the following notations:

(1) $\mathbb{K}$ is the real field or the complex field.

(2) $Mat\ r,\mathbb{K}$ is the class of all $r \times r$ matrices over the field $\mathbb{K}$.

(3) $Aut\ L$ is the class of all automorphisms of $L$.

(4) $(a_1, a_2, \ldots, a_n)$ is the diagonal matrix $diag(a_1, a_2, \ldots, a_n)$ with diagonal elements $a_1, a_2, \ldots, a_n$.

(5) $E_{ij}$ the 6-square matrix whose only non-zero entry is 1 in row $i$ column $j$.

In the first stage, we construct 8-dimensional solvable Lie algebras $L$ having 6-dimensional abelian nilradical $N(L)$. The basis of $N(L)$ is always assume to be $n_1, n_2, \ldots, n_6$. By adding to the basis $n_1, n_2, \ldots, n_6$ two elements, say $x$ and $y$, we obtain a basis $n_1, n_2, \ldots, n_6, x, y$ of $L$. Then, Lie brackets of $L$ are absolutely determined by $[x, y], [n_j, x]$ and $[n_j, y]$ for $i, j = 1,2,\ldots,6$. Since the derived algebra of a solvable Lie algebra is contained in its nilradical (Corollary 1, Lie algebras, Jacobson, 1962), these Lie brackets can be represented as follows:

$$\left[x,y\right]=\sum_{j=1}^{6}\sigma_j n_j\,,\quad \left[n_i,x\right]=\sum_{j=1}^{6}a_{ij}n_j\,,\quad \left[n_i,y\right]=\sum_{j=1}^{6}b_{ij}n_j\,,\quad 1\leq i\leq 6\,.$$

Set $A:=(a_{ij})$ and $B:=(b_{ij})$. Then, all we have to do is to determine all possibilities of the *structure constants* $\sigma_j\in\mathbb{K}$ and the *structure matrices* $A,B\in Mat\ 6,\mathbb{K}$ . The following techniques will be used.

1. First, Theorem 1 and Theorem 2 (Ndogmo & Winternitz, 1994) are used to initialize the original forms of $A$ and $B$. Moreover, we have two matrices $A$ and $B$ commute, i.e. $\left[A,B\right]=0$.

2. Next, the following two transformations simplify the structure constants.

(i) The translational transformation changes $x$ and $y$

$$x'=x+\sum_{j=1}^{6}k_j n_j\,,\qquad y'=y+\sum_{j=1}^{6}h_j n_j\,.$$

(ii) Then, we choose $k_j$ và $h_j$ appropriately to eliminate $\sigma_j$.

3. Finally, the following transformations simplify $A$ and $B$.

(i) The Kravchuk signature transforms $A$ and $B$ into Kravchuk's normal (Ndogmo & Winternitz, 1994).

(ii) We use automorphisms of $N(L)$: if $P\in Aut\ N(L)$ then $A$ and $B$ are transformed respectively into $A'=PAP^{-1}$, $B'=PBP^{-1}$.

Next, we will classify 8-dimensional solvable Lie algebras having 6-dimensional abelian nilradicals in Section 3 below.

## 3. Classification of 8-dimensional solvable Lie algebras having 6-dimensional abelian nilradicals

### 3.1 The case $\mathbb{K}=\mathbb{C}$.

From Theorem 2 (Ndogmo & Winternitz, 1994), we have a partition

$$r=6=\underbrace{\underset{\geq r_2}{r_1}+\underset{\geq r_3}{r_2}+\ldots+\underset{\geq 1}{r_n}}_{p}+\underbrace{\underset{\geq r_{p+2}}{r_{p+1}}+\underset{\geq r_{p+3}}{r_{p+2}}+\ldots+\underset{\geq 0}{r_{p+q}}}_{q}\quad \text{with } 2\leq p\leq 6,$$

$0\leq q\leq 4$.

In order to obtain all complex solvable Lie algebras of the considered type, we must consider all partitions of $r=6$. We have the following table

**Table 1. Partition of $r=6$ over $\mathbb{C}$**

| Case | $r_1+r_2+\ldots+r_p$ | $r_{p+1}+r_{p+2}+\ldots+r_{p+q}$ | $p$ | $q$ |
|---|---|---|---|---|
| 1 | $5+1$ | | 2 | 0 |
| 2 | $4+2$ | | 2 | 0 |
| 3 | $4+1+1$ | | 3 | 0 |
| 4 | $4+1$ | $+1$ | 2 | 1 |

| | | | | |
|---|---|---|---|---|
| 5 | $3+3$ | | 2 | 0 |
| 6 | $3+2+1$ | | 3 | 0 |
| 7 | $3+2$ | $+1$ | 2 | 1 |
| 8 | $3+1+1+1$ | | 4 | 0 |
| 9 | $3+1+1$ | $+1$ | 3 | 1 |
| 10 | $3+1$ | $+1+1$ | 2 | 2 |
| 11 | $2+2+2$ | | 3 | 0 |
| 12 | $2+2$ | $+2$ | 2 | 1 |
| 13 | $2+2+1+1$ | | 2 | 1 |
| 14 | $2+2+1$ | $+1$ | 2 | 1 |
| 15 | $2+2$ | $+1+1$ | 2 | 1 |
| 16 | $2+1+1+1+1$ | | 5 | 0 |
| 17 | $2+1+1+1$ | $+1$ | 4 | 1 |
| 18 | $2+1+1$ | $+1+1$ | 3 | 2 |
| 19 | $2+1$ | $+1+1+1$ | 2 | 3 |
| 20 | $1+1+1+1+1+1$ | | 6 | 0 |
| 21 | $1+1+1+1+1$ | $+1$ | 5 | 1 |
| 22 | $1+1+1+1$ | $+1+1$ | 4 | 2 |
| 23 | $1+1+1$ | $+1+1+1$ | 3 | 3 |
| 24 | $1+1$ | $+1+1+1+1$ | 2 | 4 |

Now, we will present the classification of solvable Lie algebra in the first case in Table 1. The remaining cases are done similarly. From Theorem 2 (Ndogmo & Winternitz, 1994), we have the original basic forms of the structure matrices as follows

$$A=\begin{bmatrix}1 & & & & & \\ a_1 & 1 & & & & \\ b_1 & c_1 & 1 & & & \\ d_1 & e_1 & f_1 & 1 & & \\ g_1 & h_1 & i_1 & j_1 & 1 & \\ & & & & & 0\end{bmatrix},\qquad B=\begin{bmatrix}0 & & & & & \\ a_2 & 0 & & & & \\ b_2 & c_2 & 0 & & & \\ d_2 & e_2 & f_2 & 0 & & \\ g_2 & h_2 & i_2 & j_2 & 0 & \\ & & & & & 1\end{bmatrix}.$$

From Theorem 1 (Ndogmo & Winternitz, 1994), the fact that

$$\left[A,B\right]=AB-BA=0$$

yields the following conditions of the coefficients

$$\begin{cases} c_1a_2-c_2a_1=0\\ f_1c_2-f_2c_1=0\\ j_1f_2-j_2f_1=0\\ e_1a_2+f_1b_2-e_2a_1-f_2b_1=0\\ i_1c_2+j_1e_2-i_2c_1-j_2e_1=0\\ h_1a_2+i_1b_2+j_1d_2-h_2a_1-i_2b_1-j_2d_1=0.\end{cases}$$

Suppose $\left[x, y\right] = \sum_{j=1}^{6} \sigma_j n_j, \sigma_j \in \mathbb{C}$. To eliminate $k_j$ và $h_j$, we change

$x' = x + \sum_{j=1}^{6} k_j n_j$ , $y' = y + \sum_{j=1}^{6} h_j n_j$ then $\sum_{j=1}^{6} \sigma_j n_j = \sum_{j=1}^{6} \lambda_j n_j$, with

$$\begin{cases} \lambda_1 = q_1 + k_1 \ 1 + a_1 + b_1 + d_1 + g_1 \ - h_1 \ a_2 + b_2 + d_2 + g_2 \\ \lambda_2 = q_2 + k_2 \ 1 + c_1 + e_1 + h_1 \ - h_2 \ c_2 + e_2 + h_2 \\ \lambda_3 = q_3 + k_3 \ 1 + f_1 + i_1 \ - h_3 \ f_2 + i_2 \\ \lambda_4 = q_4 + k_4 \left(1 + f_1\right) - h_4 j_2 \\ \lambda_5 = q_5 + k_5 \\ \lambda_6 = q_6 - h_6. \end{cases}$$

Without loss of generality, we can choose $k_j$ và $h_j$ to eliminate the structure constants $\sigma_j$ such that $\left[x, y\right] = 0$.

Next, we will classify the two structure matrices. To simplify the coefficients, we have 10 Kravchuk signatures (Ndogmo & Winternitz, 1994) that can occur:

**Table 2. The Kravchuk signatures of case 1 over $\mathbb{C}$ in Table 1**

| Kravchuk signatures | Conditions |
|---|---|
| $L_{1,401}$ | $a_k = b_k = c_k = d_k = e_k = f_k = 0, k = 1,2$ |
| $L_{1,104}$ | $c_k = e_k = f_k = n_k = i_k = j_k = 0, k = 1,2$ |
| $L_{1,302}$ | $a_k = b_k = c_k = j_k = 0, k = 1,2$ |
| $L_{1,203}$ | $a_k = f_k = i_k = j_k = 0, k = 1,2$ |
| $L_{1,311}$ | $a_k = b_k = c_k = 0, k = 1,2$ |
| $L_{1,113}$ | $f_k = i_k = j_k = 0, k = 1,2$ |
| $L_{1,212}$ | $a_k = j_k = 0, k = 1,2$ |
| $L_{1,221}$ | $a_k = f_k = 0, k = 1,2$ |
| $L_{1,122}$ | $c_k = j_k = 0, k = 1,2$ |
| $L_{1,131}$ | $c_k = e_k = f_k = 0, a_k = h_k, b_k = i_k, d_k = j_k, k = 1,2$ |

Now, we will consider the first Kravchuk signature in Table 2, the remaining Kravchuk signatures are done similarly.

$$L_{1,401}: \ a_k = b_k = c_k = d_k = e_k = f_k = 0, k = 1,2 .$$

In this case, we have

$$A = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ g_1 & h_1 & i_1 & j_1 & 1 & \\ & & & & & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ g_2 & h_2 & i_2 & j_2 & 0 & \\ & & & & & 1 \end{bmatrix}.$$

To destroy or normalize the off-diagonal elements of the structure matrices $A$ and $B$, we choose

$$P_1 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \frac{g_1}{j_1} & \frac{h_1}{j_1} & \frac{i_1}{j_1} & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ \frac{g_2 j_1 - g_1 j_2}{i_2 j_1 - i_1 j_2} & \frac{h_2 j_1 - h_1 j_2}{i_2 j_1 - i_1 j_2} & & 1 & \frac{j_1 j_2}{i_2 j_1 - i_1 j_2} & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \frac{i_2 j_1 - i_1 j_2}{j_1} & & & \\ & & & j_1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, P_4 = \begin{bmatrix} & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & & & 1 \end{bmatrix}, P_5 = \begin{bmatrix} & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

Transformation $P_1, P_2, P_3, P_4, P_5$ will transform $A$ and $B$ into

$$P_5 \ P_4 \ P_3 \ P_2 \ P_1 A P_1^{-1} \ P_2^{-1} \ P_3^{-1} \ P_4^{-1} \ P_5^{-1} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & 1 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix},$$

$$P_5 \ P_4 \ P_3 \ P_2 \ P_1 B P_1^{-1} \ P_2^{-1} \ P_3^{-1} \ P_4^{-1} \ P_5^{-1} = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ 1 & & 0 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix}.$$

It creates the following Lie algebras:

$$L_{1,401}:\begin{cases} A = \ 1,1,1,1,1,0 \ + E_{32} \\ B = \ 0,0,0,0,0,1 \ + E_{31} \\ \left[x,y\right] = 0. \end{cases}$$

In a similar manner, we achieve the classification table of class all solvable Lie algebras having $6$ -dimensional abelian nilradicals over $\mathbb{C}$.

Next, we use Maple to examine the decomposability of the Lie algebras and the isomorphism between them. Table 3 below presents a complete classification of all indecomposable Lie algebras up to isomorphism.

**Table 3. Classification of all 8-dimensional complex solvable Lie algebras with 6-dimensional abelian nilradical $\mathbb{C}$.**

| $L$ | Matrix $A$ | Matrix $B$ | $\left[x,y\right]$ |
|---|---|---|---|
| $L_{1,401}$ | $1,1,1,1,1,0 \ + E_{32}$ | $0,0,0,0,0,1 \ + E_{31}$ | 0 |
| $L_{1,104}$ | $1,1,1,1,1,0 \ + E_{23}$ | $0,0,0,0,0,1 \ + E_{13}$ | 0 |
| $L_{1,302}$ | $1,1,1,1,1,0 \ + E_{42} + E_{53}$ | $0,0,0,0,0,1 \ + E_{41} + E_{51}$ $+\lambda_1 E_{42} + \lambda_2 E_{53}$ | 0 |
| $L_{1,203}$ | $1,1,1,1,1,0 \ + E_{24} + E_{35}$ | $0,0,0,0,0,1 \ + E_{14} + E_{15}$ $+\lambda_1 E_{24} + \lambda_2 E_{35}$ | 0 |
| $L_{1,311}$ | $1,1,1,1,1,0 \ + E_{32} + E_{43}$ | $0,0,0,0,0,1 \ + E_{41} + \lambda \ \ E_{32} + E_{43}$ | 0 |
| $L_{1,113}$ | $1,1,1,1,1,0 \ + E_{23} + E_{34}$ | $0,0,0,0,0,1 \ + E_{14} + \lambda(E_{23} + E_{34})$ | 0 |
| $L_{1,212}$ | $1,1,1,1,1,0 \ + E_{32} + E_{43}$ $+E_{51}$ | $0,0,0,0,0,1 \ + E_{52} + \lambda_1 E_{32}$ $+\lambda_1 E_{43} + \lambda_2 E_{41} + \lambda_3 E_{51}$ | 0 |
| $L_{1,221}$ | $1,1,1,1,1,0 \ + E_{31} + E_{42}$ $+E_{54}$ | $0,0,0,0,0,1 \ + E_{53} + \lambda_1 E_{31}$ $+\lambda_2 E_{32} + \lambda_3 E_{41} + \lambda_4 E_{42} + \lambda_5 E_{54}$ | 0 |
| $L_{1,122}$ | $1,1,1,1,1,0 \ + E_{13} + E_{24}$ $+E_{45}$ | $0,0,0,0,0,1 \ + E_{35} + \lambda_1 E_{13}$ $+\lambda_2 E_{23} + \lambda_3 E_{14} + \lambda_4 E_{24} + \lambda_5 E_{45}$ | 0 |
| $L_{1,131}$ | $1,1,1,1,1,0 \ + E_{41} + E_{54}$ | $0,0,0,0,0,1 \ + E_{53} + \lambda_1 E_{31}$ $+\lambda_2 E_{41} + \lambda_3 E_{54}$ | 0 |
| $L_{2,301}$ | $1,1,1,1,0,0 \ + E_{43} + E_{65}$ | $0,0,0,0,1,1 \ + E_{42} + \lambda E_{65}$ | 0 |
| $L_{2,103}$ | $1,1,1,1,0,0 \ + E_{34} + E_{56}$ | $0,0,0,0,1,1 \ + E_{24} + \lambda E_{56}$ | 0 |
| $L_{2,202}$ | $1,1,1,1,0,0 \ + E_{31} + E_{42}$ $+E_{65}$ | $0,0,0,0,1,1 \ + E_{32} + \lambda_1 E_{31}$ $+\lambda_2 E_{41} + \lambda_3 E_{42} + \lambda_4 E_{65}$ | 0 |

| | | | |
|---|---|---|---|
| $L_{2,211}$ | $1,1,1,1,0,0\ + E_{32}$ $+E_{43} + E_{65}$ | $0,0,0,0,1,1\ + E_{41}$ $+\lambda_1\ E_{32} + E_{43}\ + \lambda_2 E_{65}$ | 0 |
| $L_{2,112}$ | $diag\ 1,1,1,1,0,0$ $+E_{23} + E_{34} + E_{56}$ | $0,0,0,0,1,1\ + E_{14} + \lambda_1 E_{23}$ $+\lambda_1 E_{34} + \lambda_2 E_{56}$ | 0 |
| $L_{2,121}$ | $1,1,1,1,0,0\ + E_{31}$ $+E_{43} + E_{65}$ | $0,0,0,0,1,1\ + E_{42} + \lambda_1 E_{21}$ $+\lambda_2 E_{31} + \lambda_3 E_{43} + \lambda_4 E_{65}$ | 0 |
| $L_{3,301}$ | $1,1,1,1,0,\alpha\ + E_{43}$ | $0,0,0,0,1,\beta\ + E_{42}$ | 0 |
| $L_{3,103}$ | $1,1,1,1,0,\alpha\ + E_{34}$ | $0,0,0,0,1,\beta\ + E_{24}$ | 0 |
| $L_{3,202}$ | $1,1,1,1,0,\alpha\ + E_{31} + E_{42}$ | $0,0,0,0,1,\beta\ + E_{32} + \lambda_1 E_{31}$ $+\lambda_2 E_{41} + \lambda_3 E_{42}$ | 0 |
| $L_{3,211}$ | $1,1,1,1,0,\alpha\ + E_{32} + E_{43}$ | $0,0,0,0,1,\beta\ + E_{41} + \lambda\ E_{32} + E_{43}$ | 0 |
| $L_{3,112}$ | $1,1,1,1,0,\alpha\ + E_{23} + E_{34}$ $+E_{56}$ | $0,0,0,0,1,\beta\ + E_{14} + \lambda\ E_{23} + E_{34}$ | 0 |
| $L_{3,121}$ | $1,1,1,1,0,\alpha\ + E_{31} + E_{43}$ | $0,0,0,0,1,\beta\ + E_{42} + \lambda_1 E_{21}$ $+\lambda_2 E_{31} + \lambda_3 E_{43}$ | 0 |
| $L_{5,201,201}$ | $1,1,1,0,0,0\ + E_{32} + E_{65}$ | $0,0,0,1,1,1\ + E_{31} + E_{64}$ | 0 |
| $L_{5,201,102}$ | $0,0,0,1,1,1\ + E_{32} + E_{56}$ | $0,0,0,1,1,1\ + E_{31} + E_{46}$ | 0 |
| $L_{5,102,201}$ | $1,1,1,0,0,0\ + E_{23} + E_{65}$ | $0,0,0,1,1,1\ + E_{13} + E_{64}$ | 0 |
| $L_{5,102,102}$ | $1,1,1,0,0,0\ + E_{23} + E_{56}$ | $0,0,0,1,1,1\ + E_{13} + E_{46}$ | 0 |
| $L_{5,201,111}$ | $1,1,1,0,0,0\ + E_{32} + E_{54}$ $+E_{65}$ | $0,0,0,1,1,1\ + E_{13}$ $+\lambda_1\ E_{54} + E_{65}\ + \lambda_2 E_{64}$ | 0 |
| $L_{5,102,111}$ | $1,1,1,0,0,0\ + E_{23}$ $+E_{54} + E_{65}$ | $0,0,0,1,1,1\ + E_{13}$ $+\lambda_1\ E_{54} + E_{65}\ + \lambda_2 E_{64}$ | 0 |
| $L_{5,111,111}$ | $1,1,1,0,0,0\ + E_{21}$ $+E_{32} + E_{54} + E_{65}$ | $0,0,0,1,1,1\ + \lambda_1 E_{21} + \lambda_1 E_{32}$ $+\lambda_2 E_{31} + \lambda_3\ E_{54} + E_{65}\ + \lambda_4 E_{64}$ | 0 |
| $L_{5,111,201}$ | $1,1,1,0,0,0\ + E_{21}$ $+E_{32} + E_{65}$ | $0,0,0,1,1,1\ + \lambda_1 E_{21}$ $+\lambda_1 E_{32} + \lambda_2 E_{31} + E_{64}$ | 0 |
| $L_{5,111,102}$ | $1,1,1,0,0,0\ + E_{21}$ $+E_{32} + E_{56}$ | $0,0,0,1,1,1\ + \lambda_1 E_{21}$ $+\lambda_1 E_{32} + \lambda_2 E_{31} + E_{46}$ | 0 |
| $L_{10,201}$ | $1,1,1,0,0,0\ + E_{32} + E_{65}$ | $0,0,0,1,0,0\ + E_{31} + \lambda E_{65}$ | $\sigma n_5 + \mu n_6$ |
| $L_{10,102}$ | $1,1,1,0,0,0\ + E_{23} + E_{56}$ | $0,0,0,1,0,0\ + E_{13} + \lambda E_{56}$ | $\sigma n_5 + \mu n_6$ |

### 3.2. The case $\mathbb{K}=\mathbb{R}$

All of the above cases also occur for $\mathbb{K}=\mathbb{R}$, but further ones also exist. From Theorem 2 (Ndogmo & Winternitz, 1994), we have partition

$$6=\underbrace{\underset{\geq r_2}{r_1}+\underset{r\geq r_3}{r_2}+\ldots+\underset{\geq 1}{r_n}}_{p}+\underbrace{\underbrace{2+2+\ldots+2}_{q}}_{2q}+\underbrace{\underset{\geq r_{p+q+2}}{r_{p+q+1}}+\underset{\geq r_{p+q+3}}{r_{p+q+2}}+\ldots+\underset{\geq 0}{r_{p+q+s}}}_{s}$$

with $p\geq 1,\ q\geq 1,\ p+2q\geq 2,\ s\geq 0.$

The above condition is equivalent to $1\leq p\leq 4, 1\leq q\leq 2,\ s\geq 0$.

In order to obtain all real solvable Lie algebras of the considered type, we must consider all partitions of $r=6$ as above. We have the following partition table

**Table 4. Partition of $r=6$ over $\mathbb{R}$**

| Case | $r_1+r_2+\ldots+r_p$ | $+2q$ | $r_{p+1}+r_{p+2}+\ldots+r_{p+q}$ | $p$ | $q$ | $s$ |
|---|---|---|---|---|---|---|
| 1 | $4$ | $+2$ | | 1 | 1 | 0 |
| 2 | $3+1$ | $+2$ | | 2 | 1 | 0 |
| 3 | $3$ | $+2$ | $+1$ | 1 | 1 | 1 |
| 4 | $2+2$ | $+2$ | | 2 | 1 | 0 |
| 5 | $2$ | $+2$ | $+2$ | 1 | 1 | 1 |
| 6 | $2+1+1$ | $+2$ | | 3 | 1 | 0 |
| 7 | $2+1$ | $+2$ | $+1$ | 2 | 1 | 1 |
| 8 | $2$ | $+2$ | $+1+1$ | 1 | 2 | 2 |
| 9 | $1+1+1+1$ | $+2$ | | 4 | 2 | 0 |
| 10 | $1+1+1$ | $+2$ | $+1$ | 3 | 1 | 1 |
| 11 | $1+1$ | $+2$ | $+1+1$ | 2 | 1 | 2 |
| 12 | $1$ | $+2$ | $+1+1+1$ | 1 | 1 | 3 |
| 13 | $1$ | $+2+2$ | | 1 | 2 | 0 |
| 14 | $1+1$ | $+2+2$ | | 2 | 2 | 0 |
| 15 | $1$ | $+2+2$ | $+1$ | 1 | 2 | 1 |

The process of finding the structure constants is similar to that over $\mathbb{C}$. Next, we will classify the two structure matrices in the case 1 (others are conducted similarly). From Theorem 2 (Ndogmo & Winternitz, 1994), we have the original forms of the structure matrices:

$$A=\begin{bmatrix}1&&&&&\\a_1&1&&&&\\b_1&c_1&1&&&\\d_1&e_1&f_1&1&&\\&&&&\alpha&\\&&&&&\alpha\end{bmatrix},\qquad B=\begin{bmatrix}0&&&&&\\a_2&0&&&&\\b_2&c_2&0&&&\\d_2&e_2&f_2&0&&\\&&&&\beta&1\\&&&&-1&\beta\end{bmatrix}.$$

According to Theorem 1 (Ndogmo & Winternitz, 1994), we have

$$\left[A,B\right]=AB-BA=0,$$

which yields the following conditions of the coefficients in the matrices:

$$\begin{cases} c_1a_2 - c_2a_1 = 0 \\ f_1c_2 - f_2c_1 = 0 \\ e_1a_2 + f_1b_2 - e_2a_1 - f_2b_1 = 0. \end{cases}$$

Next, we will classify the two structure matrices. To simplify the coefficients, we have possibilities of Kravchuk signatures (Ndogmo & Winternitz, 1994) that can occur.

**Table 5. The Kravchuk signatures of case 1 over $\mathbb{R}$ in Table 5**

| Kravchuk signatures | Conditions |
|---|---|
| $L^{1,301}$ | $a_k = b_k = c_k = 0, k = 1,2$ |
| $L^{1,103}$ | $c_k = e_k = f_k = 0, k = 1,2$ |
| $L^{1,202}$ | $a_k = f_k = 0, k = 1,2$ |
| $L^{1,211}$ | $a_1 = a_2 = 0$ |
| $L^{1,112}$ | $f_1 = f_2 = 0$ |
| $L^{1,121}$ | $c_1 = c_2 = 0$ |

Now, we will consider the first Kravchuk signature in Table 5, the remaining Kravchuk signatures are done similarly.

(301) $a_k = b_k = c_k = 0, k = 1,2$ . In this case, we have

$$A = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ d_1 & e_1 & f_1 & 1 & & \\ & & & & \alpha & \\ & & & & & \alpha \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ d_2 & e_2 & f_2 & 0 & & \\ & & & & \beta & 1 \\ & & & & -1 & \beta \end{bmatrix}.$$

To destroy or normalize off-diagonal elements of $A$ and $B,$ we choose

$$P_1 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ \frac{d_1}{f_1} & \frac{e_1}{f_1} & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & & & & & \\ \frac{d_2f_1 - d_1f_2}{e_2f_1 - e_1f_2} & 1 & \frac{f_1f_2}{e_2f_1 - e_1f_2} & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 1 & & & & & \\ & \frac{e_2f_1 - e_1f_2}{f_1} & & & & \\ & & f & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}.$$

Transformation $P_1, P_2, P_3,$ will transform $A$ and $B$ into

$$P_3\ P_2\ P_1AP_1^{-1}\ P_2^{-1}\ P_3^{-1} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & 1 & 1 & & \\ & & & & \alpha & \\ & & & & & \alpha \end{bmatrix},$$

$$P_3\ P_2\ P_1BP_1^{-1}\ P_2^{-1}\ P_3^{-1} = \begin{bmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & 1 & & 0 & & \\ & & & & \beta & 1 \\ & & & & -1 & \beta \end{bmatrix}.$$

It creates the following Lie algebras:

$$L^{1,301}: \begin{cases} A = (1,1,1,1,\alpha,\alpha) + E_{43} \\ B = (0,0,0,0,\beta,\beta) + E_{42} + E_{56} - E_{56} \\ \left[x, y\right] = 0. \end{cases}$$

The other cases can be done similarly to obtain a classification of 8-dimensional real solvable Lie algebras having 6-dimensional abelian nilradicals.

Next, we use Maple to examine the decomposability of the Lie algebras and the isomorphism between them. Table 7 below presents a complete classification of all indecomposable Lie algebras up to isomorphism.

**Table 6. Classification of all 8-dimensional real solvable Lie algebras with 6-dimensional abelian nilradical $\mathbb{R}$.**

| $L$ | Matrix $A$ | Matrix $B$ | $\left[x, y\right]$ |
|---|---|---|---|
| $L^{1,301}$ | $(1,1,1,1,\alpha,\alpha) + E_{43}$ | $(0,0,0,0,\beta,\beta) + E_{42} + E_{56} - E_{65}$ | 0 |
| $L^{1,103}$ | $(1,1,1,1,\alpha,\alpha) + E_{34}$ | $(0,0,0,0,\beta,\beta) + E_{24} + E_{56} - E_{65}$ | 0 |
| $L^{1,211}$ | $(1,1,1,1,\alpha,\alpha) + E_{43}$ | $(0,0,0,0,\beta,\beta) + E_{42} + E_{56}$ <br> $-E_{65} + k(E_{23} + E_{43})$ | 0 |
| $L^{1,112}$ | $(1,1,1,1,\alpha,\alpha) + E_{34}$ | $(0,0,0,0,\beta,\beta) + E_{24} + E_{65}$ <br> $-E_{56} + k(E_{32} + E_{34})$ | 0 |
| $L^{1,121}$ | $(1,1,1,1,\alpha,\alpha) + E_{31}$ | $(0,0,0,0,\beta,\beta) + E_{42} + E_{65}$ <br> $-E_{56} + k_1E_{21} + k_2E_{31} + k_3E_{43}$ | 0 |

By combining the results from Tables 3 and 6, we obtain the following theorem.

**Theorem 3.3**. The class of 8-dimensional solvable Lie algebras having 6-dimensional abelian nilradicals consists of 33 families of complex Lie algebras and 38 families of real ones.

**Acknowledgments:** This work was supported by a project of SPD2024.02.43.

**References**

Bianchi, L. (1903). *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*. Pisa: E. Spoerri.

Cartan, E. (1894). *Sur la structure des groupes de transformations finis et continus.* Faculty of Science, University of Paris, Academy of Paris.

Gantmacher, F. R. (1939). On the classification of real simple Lie groups. *Sbornik Mathematics*, *5*, 217–250.

Kruchkovich, G. I. (1954). Classification of three-dimensional Riemannian spaces according to groups of motions. *Uspekhi Matematicheskikh Nauk*, *9*(1), 3–40.

Le, A. V., Nguyen, A. T., Nguyen, T. C. T., Nguyen, T. M. T., & Vo, N. T. (2023). Classification of 7-dimensional solvable Lie algebras having 5-dimensional nilradicals. *Communications in Algebra*, *51*(5), 1885–1899. https://doi.org/10.1080/00927872.2022.2145300

Levi, E. E. (1905). Sulla struttura dei gruppi finiti e continui. *Atti della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematiche e Naturali*, *40*, 551–565.

Lie, M. S., & Engel, F. (1893). *Theorie der Transformationsgruppen III*. Leipzig: B. G. Teubner.

Malcev, A. I. (1945). On solvable Lie algebras. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, *9*(5), 329–356.

Mubarakzyanov, G. M. (1963). Classification of real structures of Lie algebras of fifth order. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, *3*, 99–106.

Mubarakzyanov, G. M. (1963). Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, *4*, 104–116.

Mubarakzyanov, G. M. (1963). On solvable Lie algebras. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, *1*, 114–123.

Ndogmo, J. C., & Winternitz, P. (1994). Solvable Lie algebras with Abelian nilradicals. *Journal of Physics A: Mathematical and General*, *27*, 405–423. https://doi.org/10.1088/0305-4470/27/2/024

Turkowski, P. (1990). Solvable Lie algebras of dimension six. *Journal of Mathematical Physics*, *31*(6), 1344–1350. https://doi.org/10.1063/1.528721