# CONVERGENCE OF MANN ITERATION PROCESS TO A FIXED POINT OF $(\alpha,\beta)$-NONEXPANSIVE MAPPINGS IN $L_p$ SPACES

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**Article history**

*Received: 10/03/2020; Received in revised form: 20/04/2020; Accepted: 15/05/2020*

**Abstract**

*In this paper, we prove the convergence of Mann iteration to fixed points of $(\alpha,\beta)$-nonexpansive and strictly pseudo-contractive mappings in $L_p$ spaces. In addition, by using the obtained results, we state the convergence of Mann iteration to solutions of the nonlinear integral equations.*

**Keywords:** $(\alpha,\beta)$*-nonexpansive mapping, fixed point, $L_p$ spaces, strictly pseudo-contractive mapping.*

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# SỰ HỘI TỤ CỦA DÃY LẶP MANN ĐẾN ĐIỂM BẤT ĐỘNG CỦA ÁNH XẠ $(\alpha,\beta)$-KHÔNG GIÃN TRONG KHÔNG GIAN $L_p$

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**Lịch sử bài báo**

*Ngày nhận: 10/03/2020; Ngày nhận chỉnh sửa: 20/04/2020; Ngày duyệt đăng: 15/05/2020*

**Tóm tắt**

*Trong bài báo này, chúng tôi chứng minh sự hội tụ của dãy lặp Mann đến điểm bất động của ánh xạ $(\alpha,\beta)$-không giãn và giả co chặt trong không gian $L_p$. Đồng thời, sử dụng kết quả đạt được, chúng tôi khảo sát sự hội tụ của dãy lặp Mann đến nghiệm của lớp phương trình tích phân phi tuyến.*

**Từ khóa:** *Ánh xạ $(\alpha,\beta)$-không giãn, điểm bất động, không gian $L_p$, ánh xạ giả co chặt.*

DOI: https://doi.org/10.52714/dthu.9.5.2020.814

Cite: Huynh, T. B. T., & Nguyen, T. H. (2020). Convergence of mann iteration process to a fixed point of (α,β) - nonexpansive mappings in $L_p$ spaces. Dong Thap University Journal of Science, 9(5), 26-32. https://doi.org/10.52714/dthu.9.5.2020.814.

## 1. Introduction and preliminaries

In fixed point theory, the nonexpansive mapping has received attention and been studied by many authors in several various ways. Some authors established the sufficient conditions for the existence of fixed points of nonexpansive mappings and proved some convergence results of iteration processes to fixed points and common fixed points of nonexpansive mappings. Furthermore, by constructing some inequalities which are more generalized than the inequality in the definition of a nonexpansive mapping, some authors extended a nonexpansive mapping to generalized nonexpansive mappings such as strictly pseudo-contractive mappings (Chidume, 1987), mappings satisfying condition $(C)$ (Suzuki, 2008), mappings satisfying condition $(E)$ (Garcia-Falset et al., 2011), $(\alpha)$-nonexpansive mappings (Aoyama and Kohsaka, 2011). Also, many convergence results of iteration processes to fixed points of such mappings were established. In 2018, Amini-Harandi, Fakhar and Hajisharifi introduced the generalization of a nonexpansive mapping and an $(\alpha)$-nonexpansive mapping, and is called an $(\alpha,\beta)$-nonexpansive mapping. The authors also established a sufficient condition for the existence of an approximate fixed point sequence of $(\alpha,\beta)$-nonexpansive mappings. However, the approximating fixed point of an $(\alpha,\beta)$-nonexpansive mapping by some iteration processes has not established yet. Therefore, the purpose of the current paper is to establish and prove the convergence of Mann iteration process to fixed points of $(\alpha,\beta)$-nonexpansive and strictly pseudo-contractive mappings in $L_p$ spaces.

Now, we recall some notions and lemmas found useful in what follows.

**Definition 1.1** (Amini-Harandi et al., 2018, Definition 2.2; Chidume, 1987, p. 283). Let $X$ be a normed space, $C$ be a nonempty subset of $X$ and $T : C \to C$. Then

(1) $T$ is called an $(\alpha,\beta)$-*nonexpansive mapping* if there exist $\alpha,\beta \in \mathbb{R}$ such that for all $u,v \in C$, we have

$$\begin{aligned}
&\| Tu - Tv \|^2 \\
&\le \alpha \| Tu - v \|^2 + \alpha \| Tv - u \|^2 \\
&\quad + \beta \| u - Tu \|^2 + \beta \| v - Tv \|^2 \\
&\quad + (1 - 2\alpha - 2\beta) \| u - v \|^2 .
\end{aligned}$$

(2) $T$ is called a *strictly pseudo-contractive mapping* if there exist $t > 1$ such that for all $u,v \in C$ and $r > 0$, we have

$$\| u - v \| \le \| (1+r)(u-v) - rt(Tu - Tv) \| .$$

Let $X$ be a Banach space and $X^*$ be a dual space of $X$. The normalized duality mapping $J : X \to 2^{X^*}$ defined by

$$J(u) = \quad u^* \in X^* : \| u^* \|^2 = \| u \|^2 = \left\langle u, u^* \right\rangle \quad .$$

For $p \ge 2$, we denote $E = L_p(\Omega)$ the set of measurable functions on $\Omega$ such that $| f |^p$ is Lebesgue integrable on $\Omega$. In $E = L_p(\Omega)$, the normalized duality mapping $J$ is single-valued and is denoted by $j$ (Chidume, 1987, p. 284). We shall need the following lemmas.

**Lemma 1.2** (Chidume, 1987, Lemma 1). *For* $E = L_p(\Omega)$ *and for all* $u,v \in E$, *we have*

$$\| u + v \|^2 \le (p-1) \| u \|^2 + \| v \|^2 + 2\left\langle u, j(v) \right\rangle. \quad (1.1)$$

**Lemma 1.3** (Chidume, 1987, Lemma 3). *For* $E = L_p(\Omega)$ *and let* $T : C \to C$ *be a strictly pseudo-contractive mapping with constant* $t > 1$. *Then, for all* $u,v \in E$, *we have*

$$\left\langle (I-T)u - (I-T)u, j(u-v) \right\rangle \ge \frac{t-1}{t} \| u - v \|^2 . \quad (1.2)$$

## 2. Main results

We denote $F(T) = \{ p \in C : Tp = p \}$ the set of fixed points of the mapping $T : C \to C$,

$I_1 = \{(\alpha,\beta) : \alpha < 1, \beta \le 0\}$ and $I_2 = \{(\alpha,\beta) : \alpha + 2\beta < 1, \beta > 0\}$. First, we prove that for $(\alpha,\beta) \in I_1$ or $(\alpha,\beta) \in I_2$, an $(\alpha,\beta)$-nonexpansive mapping is a quasi Lipschitz mapping, that is, there exists $L \ge 1$ such that $\|Tu - p\| \le L\|u - p\|$ for all $u \in C$ and $p \in F(T)$.

**Proposition 2.1.** *Let* $X$ *be a normed space,* $C$ *be a nonempty subset of* $X$ *and* $T : C \to C$ *be an* $(\alpha,\beta)$*-nonexpansive mapping. Then*

*(1) If* $(\alpha,\beta) \in I_1$, *then for all* $u \in C$ *and* $p \in F(T)$, *we have*

$$\|Tu - p\|^2 \le \frac{1-\alpha-2\beta}{1-\alpha}\|u - p\|^2 . \quad (2.1)$$

*(2) If* $(\alpha,\beta) \in I_2$, *then for all* $u \in C$ *and* $p \in F(T)$, *we have*

$$\|Tu - p\|^2 \le \frac{1-\alpha}{1-\alpha-2\beta}\|u - p\|^2 . \quad (2.2)$$

**Proof.** (1) For $p \in F(T)$, we have $Tp = p$. Since $T$ is an $(\alpha,\beta)$-nonexpansive mapping, for $u \in C$, we have

$$\begin{aligned}
&\|Tu - p\|^2 \\
&= \|Tu - Tp\|^2 \\
&\le \alpha\|Tu - p\|^2 + \alpha\|Tp - u\|^2 \\
&\quad + \beta\|u - Tu\|^2 + \beta\|p - Tp\|^2 \\
&\quad + (1 - 2\alpha - 2\beta)\|u - p\|^2 \\
&= \alpha\|Tu - p\|^2 + \beta\|u - Tu\|^2 \\
&\quad + (1 - \alpha - 2\beta)\|u - p\|^2 . \qquad (2.3)
\end{aligned}$$

It follows from $\beta \le 0$ and (2.3) that

$$\|Tu - p\|^2 \le \alpha\|Tu - p\|^2 + (1 - \alpha - 2\beta)\|u - p\|^2 .$$

This implies that

$$(1-\alpha)\|Tu - p\|^2 \le (1 - \alpha - 2\beta)\|u - p\|^2 .$$

By combining the above inequality with $\alpha < 1$, we get

$$\|Tu - p\|^2 \le \frac{1-\alpha-2\beta}{1-\alpha}\|u - p\|^2 .$$

(2) Since $\beta > 0$, from (2.3), we get

$$\begin{aligned}
&\|Tu - p\|^2 \\
&\le \alpha\|Tu - p\|^2 + \beta(\|u - p\| + \|p - Tu\|)^2 \\
&\quad + (1 - \alpha - 2\beta)\|u - p\|^2 \\
&\le \alpha\|Tu - p\|^2 + \beta(2\|u - p\|^2 + 2\|Tu - p\|^2) \\
&\quad + (1 - \alpha - 2\beta)\|u - p\|^2 \\
&= (\alpha + 2\beta)\|Tu - p\|^2 + (1-\alpha)\|u - p\|^2 .
\end{aligned}$$

This gives

$$(1 - \alpha - 2\beta)\|Tu - p\|^2 \le (1-\alpha)\|u - p\|^2 . \quad (2.4)$$

Since $\alpha + 2\beta < 1$, we have $1 - \alpha - 2\beta > 0$. Then, by (2.4), we get

$$\|Tu - p\|^2 \le \frac{1-\alpha}{1-\alpha-2\beta}\|u - p\|^2 . \qquad \square$$

**Remark 2.2.** *Put*

$$\delta^2 = \max\left\{\frac{1-\alpha-2\beta}{1-\alpha}, \frac{1-\alpha}{1-\alpha-2\beta}\right\}.$$

*Then, for* $(\alpha,\beta) \in I_1$ *or* $(\alpha,\beta) \in I_2$, *inequalities (2.1) and (2.2) can be rewritten in the following form: for all* $u \in C$ *and* $p \in F(T)$,

$$\|Tu - p\| \le \delta\|u - p\| . \quad (2.5)$$

Next, we prove that the set of fixed points of $(\alpha,\beta)$-nonexpansive mappings with $(\alpha,\beta) \in I_1$ or $(\alpha,\beta) \in I_2$ is closed.

**Proposition 2.3.** *Let* $X$ *be a normed space,* $C$ *be a nonempty subset of* $X$ *and* $T : C \to C$ *be an* $(\alpha,\beta)$*-nonexpansive mapping with* $(\alpha,\beta) \in I_1$ *or* $(\alpha,\beta) \in I_2$. *Then* $F(T)$ *is closed.*

**Proof.** Let $\{p_n\}$ be a sequence in $F(T)$ such that $\{p_n\}$ converges to $p \in C$. We prove that $p \in F(T)$. Since $T$ is an $(\alpha,\beta)$-nonexpansive mapping with $(\alpha,\beta) \in I_1$ or $(\alpha,\beta) \in I_2$, by using inequality (2.5), we have

$$\|Tp - p_n\| \le \delta\|p - p_n\| . \quad (2.6)$$

Taking the limit in (2.6) as $n \to \infty$ and using $\lim_{n\to\infty}\|p - p_n\| = 0$, we have

$$\lim_{n\to\infty}\|Tp - p_n\| = 0,$$

that is, the sequence $\{p_n\}$ converges to $Tp$. By combining this with the convergence of the sequence $\{p_n\}$ to $p$, we obtain $Tp = p$, that is, $p \in F(T)$. This implies that $F(T)$ is closed. $\square$

Next, we establish and prove the convergence of Mann iteration process to fixed points of $(\alpha, \beta)$-nonexpansive and strictly pseudo-contractive mappings with $(\alpha, \beta) \in I_1$ or $(\alpha, \beta) \in I_2$ in $E = L_p$ spaces.

**Theorem 2.4.** *Suppose that*

(1) $C$ is a nonempty convex subset of E.

(2) $T : C \to C$ is an $(\alpha, \beta)$-nonexpansive mapping with $(\alpha, \beta) \in I_1$ or $(\alpha, \beta) \in I_2$ and strictly pseudo-contractive mapping with constant $t > 1$ such that $F(T) \neq \varnothing$.

(3) $\{u_n\}$ is the sequence generated by $u_1 \in C$, $u_{n+1} = (1 - a_n)u_n + a_n T u_n$ *with* $n \geq 1$, *where the sequence* $\{a_n\}$ *satisfies*

$$0 \leq a_n \leq \lambda[(p-1)\delta^2 + 2\lambda - 1]^{-1}$$

*with* $\lambda = \dfrac{t-1}{t} \in (0,1)$, $\sum_{n=1}^{\infty} a_n = \infty$ *and the constant* $\delta$ *defined by (2.5).*

*Then the sequence* $\{u_n\}$ *converges to fixed points of* $T$.

**Proof**. Let $u$ be a fixed point of $T$. Since $T$ is an $(\alpha, \beta)$-nonexpansive mapping, by using the inequality (1.1), we have

$$\begin{aligned}
&\| u_{n+1} - u \|^2 \\
&= \| (1 - a_n)u_n + a_n T u_n - u \|^2 \\
&= \| a_n (T u_n - u) + (1 - a_n)(u_n - u) \|^2 \\
&\leq (p-1)a_n^2 \| T u_n - u \|^2 + (1 - a_n)^2 \| u_n - u \|^2 \\
&\qquad + 2a_n(1 - a_n)\left\langle T u_n - u, j(u_n - u)\right\rangle \\
&\leq (p-1)\delta^2 a_n^2 \| u_n - u \|^2 + (1 - a_n)^2 \| u_n - u \|^2 \\
&\qquad + 2a_n(1 - a_n)\left\langle T u_n - u, j(u_n - u)\right\rangle.
\end{aligned}$$

Moreover, by using the inequality (1.2), we find that

$$\begin{aligned}
&\left\langle T u_n - u, j(u_n - u)\right\rangle \\
&= \left\langle T u_n - u_n + u_n - u, j(u_n - u)\right\rangle \\
&= \left\langle u_n - u, j(u_n - u)\right\rangle - \left\langle u_n - T u_n, j(u_n - u)\right\rangle \\
&= \left\langle u_n - u, j(u_n - u)\right\rangle \\
&\qquad - \left\langle (I - T)u_n - (I - T)u, j(u_n - u)\right\rangle \\
&\leq \| u_n - u \|^2 - \lambda \| u_n - u \|^2 \\
&= (1 - \lambda) \| u_n - u \|^2 .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\| u_{n+1} - u \|^2 \\
&\leq (1 - a_n)^2 \| u_n - u \|^2 + (p-1)\delta^2 a_n^2 \| u_n - u \|^2 \\
&\quad + 2(1 - \lambda)a_n(1 - a_n) \| u_n - u \|^2 \\
&= [1 - 2a_n + a_n^2 + (2a_n - 2a_n^2)(1 - \lambda) \\
&\quad + (p-1)\delta^2 a_n^2] \| u_n - u \|^2 \\
&= \{1 - 2a_n\lambda + a_n^2[(p-1)\delta^2 + 2\lambda - 1]\} \| u_n - u \|^2 \\
&\leq (1 - a_n\lambda) \| u_n - u \|^2 . \qquad (2.7)
\end{aligned}$$

By using the inequality $1 - t \leq e^{-t}$ for all $t \geq 0$, from (2.7), we obtain

$$\| u_{n+1} - u \|^2 \leq e^{-a_n\lambda} \| u_n - u \|^2 \qquad (2.8)$$

for all $n \in \mathbb{N}^*$. Then, let $n$ get some values $N, N-1, \ldots, 1$ in (2.8), we find that

$$\begin{aligned}
&\| u_{N+1} - u \|^2 \\
&\leq e^{(-a_N\lambda)} \| u_N - u \|^2 \\
&\leq e^{(-a_N\lambda)}.e^{(-a_{N-1}\lambda)} \| u_{N-1} - u \|^2 \\
&\leq e^{(-a_N\lambda)}.e^{(-a_{N-1}\lambda)} \ldots\ldots\ldots e^{(-a_1\lambda)} \| u_1 - u \|^2 . \\
&= e^{\left(-\lambda\sum_{n=1}^{N} a_n\right)} \| u_1 - u \|^2 . \qquad (2.9)
\end{aligned}$$

Taking the limit in (2.9) as $N \to \infty$ and using the assumption $\sum_{n=1}^{\infty} a_n = \infty$, we conclude that the sequence $\{u_n\}$ converges to $u$. $\square$

Finally, we apply Theorem 2.4 in order to study the convergence of Mann iteration process to solutions of a nonlinear integral equation.

**Example 2.5**. $E = L_2([0,1])$ denotes a Banach space with normed

$$\| u \| = \sqrt{\int_0^1 | u(x) |^2 \, dx}.$$

Consider the following nonlinear integral equation

$$u(x) = g(x) + \int_0^1 K(x, s, u(s)) ds \qquad (2.10)$$

for all $x \in [0,1]$, where $g : E \to E$ and $K : [0,1] \times [0,1] \times E \to E$ are given mappings. Put $C = \{u \in E : u(s) \geq 0$ for all $s \in [0,1]\}$.

Then $C$ is a nonempty convex subset of $E$. For $u \in C, x \in [0,1]$, put

$$Tu(x) = g(x) + \int_0^1 K(x, s, u(s)) ds.$$

Assume that

(H1) For all $u \in C$, we have $Tu \in C$.

(H2) There exists $(\alpha, \beta) \in I_1$ or $(\alpha, \beta) \in I_2$ such that for all $x, s \in [0,1]$ and $u, v \in C$, we have

$$\begin{aligned} &| K(x,s,u(s)) - K(x,s,v(s)) |^2 \\ &\leq \alpha \, | Tu(s) - v(s) |^2 + \alpha \, | Tv(s) - u(s) |^2 \\ &\quad + \beta \, | u(s) - Tu(s) |^2 + \beta \, | v(s) - Tv(s) |^2 \\ &\quad + (1 - 2\alpha - 2\beta) \, | u(s) - v(s) |^2 . \end{aligned}$$

(H3) There exists $t > 1$ such that for all $x, s \in [0,1]$ and $u, v \in C$, we have

$$| K(x,s,u(s)) - K(x,s,v(s)) | \leq \frac{1}{t} \, | u(s) - v(s) | .$$

Consider the sequence $\{u_n\}$ defined by

$$u_1 \in C, \; u_{n+1} = (1 - a_n) u_n + a_n T u_n$$

with $n \geq 1$, where the sequence $\{a_n\}$ satisfies

$$0 \leq a_n \leq \lambda [\delta^2 + 2\lambda - 1]^{-1}$$

with $\lambda = \dfrac{t-1}{t} \in (0,1)$, $\delta^2 = \dfrac{1-\alpha}{1 - \alpha - 2\beta}$ and $\sum_{n=1}^{\infty} a_n = \infty$. Then, if the equation (2.10) has a solution $u \in C$, the sequence $\{u_n\}$ converges to $u \in C$.

**Proof.** Consider the mapping $T : C \to C$ defined by

$$Tu(x) = g(x) + \int_0^1 K(x, s, u(s)) ds$$

for all $u \in C, x \in [0,1]$. Then, by assumption (H1), we conclude that $T$ is well-defined. Note that $u \in C$ is a solution of the equation (2.10) if and only if $u \in C$ is a fixed point of $T$. Therefore, in order to prove the sequence $\{u_n\}$ converges to solution $u \in C$ of the equation (2.10), we shall prove that the sequence $\{u_n\}$ converges to $u \in F(T)$. Now, we prove that all assumptions in Theorem 2.4 are satisfied. Indeed,

(1) For all $x \in [0,1]$ and $u, v \in C$, using the inequality Holder, we find that

$$\begin{aligned} &| Tu(x) - Tv(x) | \\ &\leq \int_0^1 | K(x,s,u(s)) - K(x,s,v(s)) | ds \\ &\leq \sqrt{\int_0^1 ds} \sqrt{\int_0^1 | K(x,s,u(s)) - K(x,s,v(s)) |^2 ds} \\ &= \sqrt{\int_0^1 | K(x,s,u(s)) - K(x,s,v(s)) |^2 ds}. \qquad (2.11) \end{aligned}$$

Then, from (2.11) and using the assumption (H2), we obtain

$$\begin{aligned} &| Tu(x) - Tv(x) |^2 \\ &\leq \int_0^1 | K(x,s,u(s)) - K(x,s,v(s)) |^2 ds \\ &\leq \int_0^1 [\alpha \, | Tu(s) - v(s) |^2 + \alpha \, | Tv(s) - u(s) |^2 \\ &\quad + \beta \, | u(s) - Tu(s) |^2 + \beta \, | v(s) - Tv(s) |^2 \\ &\quad + (1 - 2\alpha - 2\beta) \, | u(s) - v(s) |^2 ] ds \\ &= \alpha \int_0^1 | Tu(s) - v(s) |^2 \, ds + \alpha \int_0^1 | Tv(s) - u(s) |^2 ds \\ &\quad + \beta \int_0^1 | u(s) - Tu(s) |^2 ds + \beta \int_0^1 | v(s) - Tv(s) |^2 ds \\ &\quad + (1 - 2\alpha - 2\beta) \int_0^1 | u(s) - v(s) |^2 \, ds \end{aligned}$$

$$= \alpha \| Tu - v \|^2 + \alpha \| Tv - u \|^2 + \beta \| u - Tu \|^2$$
$$+ \beta \| v - Tv \|^2 + (1 - 2\alpha - 2\beta) \| u - v \|^2 . \quad (2.12)$$

By taking the integral both sides of (2.12) with respect to the variable $x$ on $[0,1]$, we have

$$\int_0^1 | Tu(x) - Tv(x) |^2 dx$$
$$\leq \int_0^1 [\alpha \| Tu - v \|^2 + \alpha \| Tv - u \|^2 + \beta \| u - Tu \|^2$$
$$+ \beta \| v - Tv \|^2 + (1 - 2\alpha - 2\beta) \| u - v \|^2 ] dx$$
$$= [\alpha \| Tu - v \|^2 + \alpha \| Tv - u \|^2 + \beta \| u - Tu \|^2$$
$$+ \beta \| v - Tv \|^2 + (1 - 2\alpha - 2\beta) \| u - v \|^2 ] \int_0^1 dx.$$

This gives that

$$\| Tu - Tv \|^2$$
$$\leq \alpha \| Tu - v \|^2 + \alpha \| Tv - u \|^2 + \beta \| u - Tu \|^2 + \beta \| v - Tv \|^2$$
$$+ (1 - 2\alpha - 2\beta) \| u - v \|^2 .$$

This implies that $T$ is an $(\alpha, \beta)$-nonexpansive mapping.

(2) For all $x \in [0,1]$ and $u, v \in C$, from (2.11), we have

$$| Tu(x) - Tv(x) |^2 \leq \int_0^1 | K(x,s,u(s)) - K(x,s,v(s)) |^2 ds.$$

By combining this with the assumption (H3), there exists $t > 1$ such that

$$| Tu(x) - Tv(x) |^2$$
$$\leq \int_0^1 \frac{1}{t^2} | u(s) - v(s) |^2 ds$$
$$= \frac{1}{t^2} \| u - v \|^2 .$$

By taking the integral both sides of the above inequality with respect to the variable $x$ on $[0,1]$, we obtain

$$\int_0^1 | Tu(x) - Tv(x) |^2 dx$$
$$\leq \int_0^1 \frac{1}{t^2} \| u - v \|^2 \; dx$$
$$= \frac{1}{t^2} \| u - v \|^2 \int_0^1 dx$$
$$= \frac{1}{t^2} \| u - v \|^2 .$$

This gives that $t \| Tu - Tv \| \leq \| u - v \|$. Then, for all $r > 0$, we get

$$0 \leq r \| u - v \| - rt \| Tu - Tv \| .$$

By adding $\| u - v \|$ to both sides of the above inequality, we find that

$$\| u - v \|$$
$$\leq (r+1) \| u - v \| - rt \| Tu - Tv \|$$
$$\leq \| (r+1)(u - v) - rt(Tu - Tv) \| .$$

This implies that $T$ a strictly pseudo-contractive mapping with constant $t > 1$.

Therefore, all assumptions in Theorem 2.4 are satisfied. Thus, by Theorem 2.4, we conclude that the sequence $\{u_n\}$ converges to $u \in F(T)$ and hence the sequence $\{u_n\}$ converges to solution $u \in C$ of the nonlinear integral equation (2.10). □

The following example guarantees the existence of two mappings $g, K$ satisfying all the assumptions in Example 2.5. Also, this example illustrates the existence of the sequence $\{a_n\}$ in Theorem 2.4.

**Example 2.6.** $E = L_2([0,1])$ denotes a Banach space with normed

$$\| u \| = \sqrt{\int_0^1 | u(x) |^2 \; dx}$$

and $C = \{u \in E : u(s) \geq 0 \text{ for all } s \in [0,1]\}$. Consider the following nonlinear integral equation

$$u(x) = \frac{11}{12} x^2 + \int_0^1 \frac{(1+s^2)x^2 u(s)}{4(1+u(s))} ds \qquad (2.13)$$

for all $x \in [0,1]$, where $u \in C$ is a function which we must find out. For all $x, s \in [0,1]$ and $u \in C$, put

$$K(x,s,u(s))=\frac{(1+s^2)x^2u(s)}{4(1+u(s))}$$

and

$$Tu(x)=\frac{11}{12}x^2+\int_0^1\frac{(1+s^2)x^2u(s)}{4(1+u(s))}ds.$$

We will prove the assumptions (H1), (H2) and (H3) in Example 2.5 are satisfied. Indeed,

(1) For $u\in C$, we have $u(s)\geq 0$ for all $s\in[0,1]$. Therefore, $Tu(x)\geq 0$ for $x\in[0,1]$. Moreover, for all $x\in[0,1]$, we have

$$\begin{aligned}Tu(x)&=\frac{11}{12}x^2+\int_0^1\frac{(1+s^2)x^2u(s)}{4(1+u(s))}ds\\&\leq\frac{11}{12}x^2+x^2\int_0^1\frac{(1+s^2)}{4}ds\\&=\frac{5}{4}x^2.\end{aligned}$$

This implies that $Tu\in E$. Thus, $Tu\in C$.
(2) For all $x,s\in[0,1]$ and $u,v\in C$, we have

$$\begin{aligned}&|\,K(x,s,u(s))-K(x,s,v(s))\,|\\&\leq\frac{(1+s^2)x^2}{4}\left|\frac{u(s)}{1+u(s)}-\frac{v(s)}{1+v(s)}\right|\\&\leq\frac{1}{2}\left|u(s)-v(s)\right|.\end{aligned}\tag{2.14}$$

This proves that the assumption (H2) is satisfied with $\alpha=\frac{3}{8},\beta=0.$

(3) From (2.14), we conclude that the assumption (H3) is satisfied with $t=2.$

Therefore, the assumptions (H1), (H2) and (H3) in Example 2.5 are satisfied. Moreover, it is easy to check that $u(x)=x^2$ for all $x\in[0,1]$ is a solution to the nonlinear integral equation (2.13). Note that from $\lambda=\frac{t-1}{t}=\frac{1}{2}$ and $\delta^2=\frac{1-\alpha}{1-\alpha-2\beta}=1,$ the condition $0\leq a_n\leq\lambda[\delta^2+2\lambda-1]^{-1}$ becomes $0\leq a_n\leq\frac{1}{2}.$

By choosing $a_n=\frac{1}{2n}$ for all $n\geq 1,$ we have $\sum_{n=1}^{\infty}a_n=\infty.$ Then, by Example 2.5, the sequence $\{u_n\}$ defined by: $u_1\in C,$

$$\begin{aligned}&u_{n+1}(x)\\&=\frac{2n-1}{2n}u_n(x)+\frac{1}{2n}\left\{\frac{11}{12}x^2+\int_0^1\frac{(1+s^2)x^2u_n(s)}{4(1+u_n(s))}ds\right\}\end{aligned}$$

for all $n\in[0,1]$ and $n\geq 1$ converges to solution $u(x)=x^2$ for all $x\in[0,1]$ of the nonlinear integral equation (2.13). □

**Acknowledgments:** This research is supported by the project SPD2019.02.14./.

**References**

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