THE JACOBSON RADICAL TYPES OF LEAVITT PATH ALGEBRAS WITH COEFFICIENTS IN A COMMUTATIVE UNITAL SEMIRING

Le Hoang Mai

Department of Mathematics Teacher Education, Dong Thap University, Vietnam

Email: lhmai@dthu.edu.vn

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Abstract

In this paper, we calculate the $J-$radical and J_s-radical of the Leavitt path algebras with coefficients in a commutative semiring of some finite graphs. In particular, we calculate $J-$radical and J_s-radical of the Leavitt path algebras with coefficients in a field of acyclic graphs, no-exit graphs and give applicable examples.

Keywords: *Acyclic graph, $J-$radical of semiring; J_s- radical of semiring, Leavitt path algebra, no-exit graph.*

CÁC KIỂU CĂN JACOBSON CỦA CÁC ĐẠI SỐ ĐƯỜNG ĐI LEAVITT VỚI HỆ SỐ TRONG NỬA VÀNH CÓ ĐƠN VỊ GIAO HOÁN

Lê Hoàng Mai

Khoa Sư phạm Toán học, Trường Đại học Đồng Tháp, Việt Nam

Email: lhmai@dthu.edu.vn

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Tóm tắt

Trong bài viết này, chúng tôi tính $J-$căn và J_s-căn của đại số đường đi Leavitt với hệ số trên một nửa vành có đơn vị giao hoán của một số dạng đồ thị hữu hạn. Trong trường hợp đặc biệt, chúng tôi tính $J-$căn và J_s-căn của đại số đường đi Leavitt với hệ số trên một trường của lớp các đồ thị không chu trình, lớp các đồ thị không có lối rẽ và cho các ví dụ áp dụng.

Từ khóa: *Đồ thị không chu trình, $J-$căn của nửa vành, J_s-căn của nửa vành, đại số đường đi Leavitt, đồ thị không có lối rẽ.*

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1. Introduction

Bourne (1951) defined the $J-$radical of a hemiring based on left (right) semiregular ideals and, subsequently, Iizuka (1959) proved that this radical can be determined via irreducible semimodules. Katsov and Nam (2014) defined the J_s-radical for hemirings using simple semimodules and obtained some results on the structure of additively idempotent hemirings through this radical. Recently, Mai and Tuyen (2017) have used the concepts of $J-$radical and J_s-radical of hemiring to study the structure of some hemirings. The concepts and results related to $J-$radical and J_s-radical of hemirings can be found in Bourne (1951), Iizuka (1959), Katsov & Nam (2014), Mai & Tuyen (2017).

Given a (row-finite) directed graph E and a field $K,$ Abrams and Pino (2005) introduced the Leavitt path algebra $L_K(E)$. These Leavitt path algebras are a generalization of the Leavitt algebras $L_K(1,n)$ of Leavitt (1962). Tomforde (2011) presented a straightforward generalization of the constructions of the Leavitt path algebras $L_R(E)$ with coefficients in a unita commutative ring R and studied some fundamental properties of those algebras. Katsov et al. (2017) continued to generalize the Leavitt path algebras $L_R(E)$ with coefficients in a commutative semiring R and studied some fundamental properties, especially, they studied its ideal-simpleness and congruence-simpleness. The concepts and results relating to the Leavitt path algebras $L_K(E)$ of the graph E with K is a field, unita commutative ring or commutative semiring can be found in Abrams & Pino (2005), Tomforde (2011), Katsov et al. (2017), Abrams (2015), Nam and Phuc (2019).

In this paper, we study the $J-$radical and the J_s-radical for the Leavitt path algebras $L_R(E)$ of directed graphs E with coefficients in a commutative semiring R. Specifically, we calculate the $J-$radical and the J_s-radical for the Leavitt path algebras $L_R(E)$ with coefficients in a commutative semiring R of some finite directed graphs E. In particular, we calculate the $J-$radical and the J_s-radical for the Leavitt path algebras $L_K(E)$ with coefficients in a field K of acyclic graphs, no-exit graphs and applicable examples.

We will present the main results in Section 4. In Sections 2 and 3, we will briefly present the necessary preparation knowledge in this article.

2. $J-$radical and J_s-radical of semirings

In this section, we survey some concepts and results from previous works (Golan, 1999; Iizuka, 1959; Katsov & Nam, 2014; Mai & Tuyen, 2017) and use them in the main section of this article. First, we recall the $J-$radical and the J_s-radical concepts of hemirings.

A hemiring R is an algebra $(R,+,.,0)$ such that the following conditions are satisfied:

(a) $(R,+,.,0)$ is a commutative monoid with identity element 0;

(b) $(R,.)$ is a semigroup;

(c) Multiplication distributes over addition on either side;

(d) $r0=0=0r$ for all $r\in R.$

A hemiring R is called a *semiring* if its multiplicative semigroup $(R,.,1)$ is a monoid with identity element $1.$

Note that, if R is a ring then, it is also a hemiring; otherwise, it is not true.

A *left* $R-$*semimodule* M over a commutative hemiring R is a commutative monoid $(M,+,0_M)$ together with a scalar multiplication $(r,m)\mapsto rm$ from $R\times M$ to M

which satisfies the identities: for all $r, r' \in R$ and $m, m' \in M$:

(a) $r(m+m') = rm + rm'$;

(b) $(r+r')m = rm + r'm$;

(c) $(rr')m = r(r'm)$;

(d) $r0_M = 0_M = 0m$.

If R is a semiring with identity element $1 \neq 0$ and $1m = m$ for all $m \in M$ then M is called *unita left* $R-$*semimodule.*

An $R-$*algebra* A over a commutative semiring R is a $R-$semimodule A with an associative bilinear $R-$semimodule multiplication "." on A. An $R-$algebra A is unital if $(A,.)$ is actually a monoid with a neutral element $1_A \in A$, i.e., $a1_A = 1_A a = a$ for all $a \in A$. For example, every hemiring is an $\mathbb{N}-$algebra, where $\mathbb{N}$ is the commutative semiring of non-negative integers.

Let R be a commutative semiring and $\{x_i \mid i \in I\}$ be a set of independent, non-commuting indeterminates. Then, $R\langle x_i \mid i \in I\rangle$ will denote the free $R-$algebra generated by the indeterminates $\{x_i \mid i \in I\}$, whose elements are polynomials in the non-commuting variables $\{x_i \mid i \in I\}$ with coefficients from R that commute with each variable $\{x_i \mid i \in I\}$.

Iizuka (1959) used a class of irreducible left semimodule to characterize the $J-$radical of hemirings. A nonzero cancellative left semimodule M over a hemiring R is *irreducible* if for an arbitrarily fixed pair of elements $u, u' \in M$ with $u \neq u'$ and any $m \in M$, there exist $a, a' \in R$ such that

$$m + au + a'u' = au' + a'u.$$

Theorem 2.1. [Iizuka (1959), Theorem 8]. *Let* R *be a hemiring. Then,* $J-$*radical of hemiring* R *is*

$$J(R) = \cap\{(0:M) \mid M \in \Im\},$$

where $(0:M) = \{r \in R \mid rM = 0\}$ *is a ideal of* R *and* $\Im$ *is the class of all irreducible left* $R-$*semimodules.*

When $\Im = \phi$, $J(R) = R$ by convention. The hemiring R is said to be $J-$*semisimple* if $J(R) = 0$.

Katsov and Nam (2014) used a class of simple left $R-$semimodules to define the J_s- radical of hemirings. A left $R-$semimodule M is *simple* if the following conditions are satisfied:

(a) $RM \neq 0$;

(b) M has only two trivial subsemimodules;

(c) M has only two trivial congruences.

Let R be a hemiring, subtractive ideal $J_s(R) = \cap\{(0:M) \mid M \in \Im'\}$ is called J_s-*radical* of hemiring R, where $\Im'$ is a class of all simple left $R-$semimodules.

When $\Im' = \phi$, $J_s(R) = R$ by convention. The hemiring R is said to be J_s-*semisimple* if $J_s(R) = 0$.

Remark 2.2. If R is a hemiring and is not a ring, then generally $J(R) \neq J_s(R)$ and if R is a ring then $J(R) = J_s(R)$, it is called *the Jacobson radical* in ring theory. In particular, if K is a field then $J(K) = J_s(K) = 0$.

Theorem 2.3. [Katsov & Nam (2014), Corollary 5.11]. *For all matrix hemirings* $M_n(R), n \geq 1$, *over a hemiring* R, *the following equations hold:*

(a) $J(M_n(R)) = M_n(J(R))$;

(b) $J_s(M_n(R)) = M_n(J_s(R))$.

Theorem 2.4. [Mai & Tuyen (2017), Corollary 1]**.** *Let* R *be a hemiring and* R_1, R_2 *be its subhemirings. If* $R = R_1 \oplus R_2$, *then* $J(R) = J(R_1) \oplus J(R_2)$ *and* $J_s(R) = J_s(R_1) \oplus J_s(R_2)$.

3. The Leavitt path algebras

In this section, we survey some concepts and results from previous works (Abrams & Pino, 2005; Katsov et al., 2017; Abrams, 2015), and use them in the main section of this article. First, we recall the Leavitt path algebras having coefficients in an arbitrary commutative semiring.

A (directed) graph $E = (E^0, E^1, s, r)$ consists of two disjoint sets E^0 and E^1 - *vertices* and *edges*, respectively - and two maps $r, s : E^1 \to E^0$. If $e \in E^1$, then $s(e)$ and $r(e)$ are called the *source* and *range* of e, respectively. The graph E is *finite* if $|E^0| < +\infty$ and $|E^1| < +\infty$. A vertex $v \in E^0$ for which $s^{-1}(\mathrm{v})$ is empty is called a *sink*; and a vertex $v \in E^0$ is *regular* if $0 < |s^{-1}(\mathrm{v})| < +\infty$. In this article, we consider only finite graphs.

A *path* $p = e_1 e_2 ... e_n$ in a graph E is a sequence of edges $e_1, e_2, ..., e_n \in E^1$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, 2, ..., n-1$. In this case, we say that the path p starts at the vertex $s(p) := s(e_1)$ and ends at the vertex $r(e_n) =: r(p)$, and has *length* $|p| = n$. We consider the vertices in E^0 to be paths of length 0. If $s(p) = r(p)$, then p is a *closed path* based at $v = s(p) = r(p)$. If $c = e_1 e_2 ... e_n$ is a closed path of positive length and all vertices $s(e_1), s(e_2), ..., s(e_n)$ are distinct, then the path c is called a *cycle*. An edge f is an *exit* for a path $p = e_1 e_2 ... e_n$ if $s(f) = s(e_i)$ but $f \neq e_i$ for some $1 \le i \le n$.

A graph E is *acyclic* if it has no cycles. A graph E is said to be a *no-exit* graph if no cycle in E has an exit.

Remark 3.1. If E is a finite acyclic graph, then it is a no-exit graph, and the converse is not true in general.

Definition 3.2 [Katsov et al. (2017), Definition 2.1**].** Let $E = (E^0, E^1, s, r)$ be a graph and R be a commutative semiring. The Leavitt path algebra $L_R(E)$ of the graph E with coefficients in R is the $R-$algebra presented by the set of generators $E^0 \cup E^1 \cup (E^1)^*$ $-$where $E^1 \to (E^1)^*, e \mapsto e^*$, is a bijection with $E^0, E^1, (E^1)^*$ pairwise disjoint, satisfying the following relations:

(1) $vw = \delta_{v,w} w$ (δ is the Kronecker symbol) for all $v, w \in E^0$;

(2) $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$;

(3) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$;

(4) $v = \sum_{e \in s^{-1}(v)} ee^*$ whenever $v \in E^0$ is a regular.

The following are two structural theorems of the Leavitt path algebras over any field K of acyclic graphs, no-exit graphs and applicable examples.

Theorem 3.3 [Abrams (2015), Theorem 9]**.** *Let* E *be a finite acyclic graph and* K *any field. Let* $w_1, ..., w_t$ *denote the sinks of* E *(at least one sink must exist in any finite acyclic graph). For each* w_i, *let* n_i *denote the number of elements of path in* E *having range vertex* w_i *(this includes* w_i *itself, as a path of length 0). Then*

$$L_K(E) \cong \bigoplus_{i=1}^{t} M_{n_i}(K).$$

Example 3.4. Let K be a field and E a finite acyclic graph has form

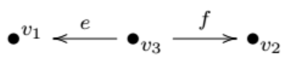

Figure 1

E has two sinks $\{v_1, v_2\}$, v_1 has two paths $\{v_1, e\}$ having range vertex v_1 and v_2 has two paths $\{v_2, f\}$ having range vertex v_2. From Theorem 3.3, we have

$$L_K(E) \cong M_2(K) \oplus M_2(K).$$

Theorem 3.5 [Nam and Phuc (2019), Corollary 2.12]**.** *Let K be a field, E a finite no-exit graph, $\{c_1, ..., c_l\}$ the set of cycles, and $\{v_1, ..., v_k\}$ the set of sinks. Then*

$$L_K(E) \cong (\bigoplus_{i=1}^{k} M_{m_i+1}(K)) \oplus (\bigoplus_{j=1}^{l} M_{n_j+1}(K[x, x^{-1}])),$$

where for each $1 \le i \le k$, m_i is the number of path ending in the sink v_i, for each $1 \le j \le l$, n_j is the number of path ending in a fixed (although arbitrary) vertex of the cycle c_j which do not contain the cycle itself and $K[x, x^{-1}]$ Laurent polynomials algebra over field K.

Example 3.6. Let K be a field and E a finite no-exit graph has form

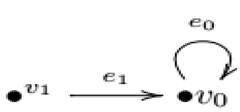

Figure 2

E has only one cycle e_0, no sink and one path e_1 other cycle e_0 having range vertex v_0. From Theorem 3.5 deduced

$$L_K(E) \cong M_2(K[x, x^{-1}]).$$

Remark 3.7. From Remark 3.1, Theorem 3.3 is a corollary of Theorem 3.5.

4. Main results

In this section, we calculate the $J-$radical and the J_s-radical for the Leavitt path algebras $L_R(E)$ with coefficients in a commutative semiring R of some finite directed graphs E. In particular, we calculate the $J-$radical and the J_s-radical for the Leavitt path algebras $L_K(E)$ with coefficients in a field K of acyclic graphs, no-exit graphs and applicable examples.

Proposition 4.1. *Let R be a commutative semiring and $E = (E^0, E^1, s, r)$ a graph has form*

Figure 3

i.e., $E^0 = \{v\}$ and $E^1 = \{e\}$. Then

$$J(L_R(E)) = J(R[x, x^{-1}]) \text{ và}$$

$$J_s(L_R(E)) = J_s(R[x, x^{-1}]),$$

where $R[x, x^{-1}]$ is a Laurent polynomials algebra over semiring R.

Proof. It is well known that $L_R(E) = R\langle v, e, e^* \rangle$ is a Leavitt path algebra generated by set $\{v, e, e^*\}$ and *Laurent polynomials algebra* $R[x, x^{-1}]$ generated by set $\{x, x^{-1}\}$. Consider the map

$$f : L_R(E) \to R[x, x^{-1}]$$

determined by $f(v) = 1$, $f(e) = x$ and $f(e^*) = x^{-1}$. Then, it is easy to check that f is an algebraic isomorphism, i.e.,

$$L_R(E) \cong R[x, x^{-1}],$$

the proof is completed. □

Proposition 4.2. Let R be a commutative semiring and $E = (E^0, E^1, s, r)$ a graph has form

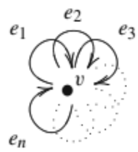

Figure 4

i.e., $E^0 = \{v\}$ *and* $E^1 = \{e_1, ..., e_n\}$ *with* $n \geq 1$. *Then*

$$J(L_R(E)) = J(L_{1,n}(R)) \text{ and}$$
$$J_s(L_R(E)) = J_s(L_{1,n}(R)),$$

where $L_{1,n}(R)$ *is a Leavitt algrbra type* $(1, n)$.

Proof. It is well known that $L_R(E) = R\langle v, e_1, ..., e_n, e_1^*, ..., e_n^*\rangle$ is a Leavitt path algebra generated by set $\{v, e_1, ..., e_n, e_1^*, ..., e_n^*\}$ and $L_{1,n}(R) = R\langle x_1, ..., x_n, y_1, ..., y_n\rangle$, where $x_i y_j = \delta_{ij}$ and $\sum_{i=1}^{n} x_i y_i = 1$ for $1 \leq i, j \leq n$, is a Leavitt algebra type $(1, n)$. Consider the map

$$f : L_R(E) \to L_{1,n}(R)$$

Determined by $f(v) = 1$, $f(e_i) = x_i$ and $f(e_i^*) = y_i$ for each $1 \leq i \leq n$. Then, it is easy to check that f is an algebraic isomorphism, i.e., $L_R(E) \cong L_{1,n}(R)$, the proof is completed. □

Proposition 4.3. *Let* R *be a commutative semiring and* $E = (E^0, E^1, s, r)$ *a graph has form*

Figure 5

i.e., $E^0 = \{v_1, ..., v_n\}$ *and* $E^1 = \{e_1, ..., e_{n-1}\}$ *with* $n \geq 2$. *Then*

$$J(L_R(E)) = M_n(J(R)) \text{ và } J_s(L_R(E)) = M_n(J_s(R)),$$

where $M_n(R)$ *is a matrix algebra over semiring* R.

Proof. It is well-known that $L_R(E) = R\langle v_1, ..., v_n, e_1, ..., e_{n-1}, e_1^*, ..., e_{n-1}^*\rangle$ is a Leavitt path algebra generated by set $\{v_1, ..., v_n, e_1, ..., e_{n-1}, e_1^*, ..., e_{n-1}^*\}$ and

$$M_n(R) = R\langle E_{i,j} \mid 1 \leq i, j \leq n\rangle,$$

is a matrix algebra generated by set $\{E_{i,j} \mid 1 \leq i, j \leq n\}$, where $E_{i,j}$ are the standard elementary matrices in the matrix semiring $M_n(R)$. Consider the map

$$f : L_R(E) \to M_n(R)$$

determined by $f(v_i) = E_{i,i}$, $f(e_i) = E_{i,i+1}$ and $f(e_i^*) = E_{i+1,i}$ for each $1 \leq i \leq n$. Then, it is easy to check that f is an algebraic isomorphism, i.e., $L_R(E) \cong M_n(R)$. Thence inferred $J(L_R(E)) = J(M_n(R))$ *and* $J_s(L_R(E)) = J_s(M_n(R))$. From Theorem 2.3, the proof is completed. □

Proposition 4.4. *Let* R *be a commutative semiring and* $E = (E^0, E^1, s, r)$ *a graph has form*

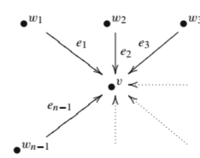

Figure 6

i.e., $E^0 = \{v, w_1, ..., w_{n-1}\}$ *and* $E^1 = \{e_1, ..., e_{n-1}\}$ *with* $n \geq 2$. *Then* $J(L_R(E)) = M_n(J(R))$ *and* $J_s(L_R(E)) = M_n(J_s(R))$, *where* $M_n(R)$ *is a matrix algebra over semiring* R.

Proof. It is well-known that $L_R(E) = R\langle v, w_1, ..., w_{n-1}, e_1, ..., e_{n-1}, e_1^*, ..., e_{n-1}^*\rangle$ is a Leavitt path algebra generated by set $\{v, w_1, ..., w_{n-1}, e_1, ..., e_{n-1}, e_1^*, ..., e_{n-1}^*\}$. Consider the map

$$f : L_R(E) \to M_n(R)$$

determined by $f(v) = E_{1,1}$, $f(w_i) = E_{i+1,i+1}$, $f(e_i) = E_{i,n}$ and $f(e_i^*) = E_{n,i}$ for each $1 \leq i \leq n-1$. Then, it is easy to check that f is an algebraic isomorphism, i.e., $L_R(E) \cong M_n(R)$. Thence it infers

$J(L_R(E)) = J(M_n(R))$ *and* $J_s(L_R(E)) = J_s(M_n(R)).$

From Theorem 2.3, the proof is completed. □

Corollary 4.5. *Let* R *be a commutative semiring and* $E = (E^0, E^1, s, r)$ *a graph has form Figure 5 or Figure 6. Then*

(a) If $R = \mathbb{N}$ *then* $J(L_{\mathbb{N}}(E)) = J_s(L_{\mathbb{N}}(E)) = 0,$ *where* $\mathbb{N}$ *is the commutative semiring of non-negative integers.*

(b) If R *be a unita commutative ring, then* $J(L_R(E)) = J_s(L_R(E)) = M_n(J(R)),$ *where* $J(R)$ *is a Jacobson radical of ring* $R.$

(c) If K *is a field, then* $J(L_K(E)) = J_s(L_K(E)) = 0.$

Proof. (a) According to Lemma 5.10 of Katsov & Nam (2014), $J(\mathbb{N}) = J_s(\mathbb{N}) = 0.$

(b) Since R is a ring, $J(R) = J_s(R).$

(c) Since K is a field,

$$J(K) = J_s(K) = 0.$$

From Proposition 4.3 or Proposition 4.4, the proof is completed. □

Theorem 4.6. *Let* K *be an any field,* E *a finite no-exit graph,* $\{c_1,...,c_l\}$ *the set of cycles, and* $\{v_1,...,v_k\}$ *the set of sinks. Then*

(a) $$J(L_K(E)) = \bigoplus_{j=1}^{l} M_{n_j+1}(J(K[x,x^{-1}])),$$

(b) $$J_s(L_K(E)) = \bigoplus_{j=1}^{l} M_{n_j+1}(J_s(K[x,x^{-1}])),$$

where for each $1 \le j \le l,$ n_j *is the number of path ending in a fixed (although arbitrary) vertex of the cycle* c_j *which do not contain the cycle itself and* $K[x,x^{-1}]$ *Laurent polynomial algebra over field* $K.$

Proof. From Theorem 3.5, we have

$$L_K(E) \cong (\bigoplus_{i=1}^{k} M_{m_i+1}(K)) \oplus (\bigoplus_{j=1}^{l} M_{n_j+1}(K[x,x^{-1}])),$$

where $\{c_1,...,c_l\}$ *the set of cycles, and* $\{v_1,...,v_k\}$ *the set of sinks for each* $1 \le i \le k,$ m_i *is of path ending in the sink* $v_i,$ *for each* $1 \le j \le l,$ n_j *is the number of path ending in a fixed (although arbitrary) vertex of the cycle* c_j *which do not contain the cycle itself.*

From Theorem 2.4, we have

$$J(L_K(E)) = (\bigoplus_{i=1}^{k} J(M_{m_i+1}(K))) \oplus (\bigoplus_{j=1}^{l} J(M_{n_j+1}(K[x,x^{-1}]))),$$

$$J_s(L_K(E)) = (\bigoplus_{i=1}^{k} J_s(M_{m_i+1}(K))) \oplus (\bigoplus_{j=1}^{l} J_s(M_{n_j+1}(K[x,x^{-1}]))).$$

From Theorem 2.3, we have

$$J(L_K(E)) = (\bigoplus_{i=1}^{k} M_{m_i+1}(J(K))) \oplus (\bigoplus_{j=1}^{l} M_{n_j+1}(J(K[x,x^{-1}]))),$$

$$J_s(L_K(E)) = (\bigoplus_{i=1}^{k} M_{m_i+1}(J_s(K))) \oplus (\bigoplus_{j=1}^{l} M_{n_j+1}(J_s(K[x,x^{-1}]))).$$

From K is a field and Remark 2.2, we have $J(K) = J_s(K) = 0,$ the proof is completed. □

Example 4.7. (a) Let K be field and E a graph has form Figure 3. Since graph E in Figure 3 is no-exit, there exists only one cycle $e,$ no sink and not path other cycle e having *ending in vertex* $v.$ From Theorem 4.6, we have $J(L_K(E)) = J(K[x,x^{-1}])$ and

$$J_s(L_K(E)) = J_s(K[x,x^{-1}]).$$

This result is also the result in Proposition 4.1 when the commutative semiring R is a field.

(b) Let K be a field and E a graph has form Figure 4. Since graph E in Figure 4 is no-exit, there is n cycles e_j for each $1 \le j \le n,$ no sink and for each $1 \le j \le n,$ has $n-1$ paths other cycle e_j having ending vertex v in cycle $e_j.$ From Theorem 4.6, we have $J(L_K(E)) = M_n(J(K[x,x^{-1}])) \oplus ... \oplus M_n(J(K[x,x^{-1}])),$ $J_s(L_K(E)) = M_n(J_s(K[x,x^{-1}])) \oplus ... \oplus M_n(J_s(K[x,x^{-1}])),$ the directed sum of the right hand side has n terms. This result is also the result in Proposition 4.2 when the commutative semiring R is a field, because

$$L_{1,n}(K) \cong M_n(K[x,x^{-1}]) \oplus ... \oplus M_n(K[x,x^{-1}]).$$

(c) Let K be a field and E be a no-exit graph has form Figure 2. From Theorem 4.6, we have $J(L_K(E)) = M_2(J(K[x, x^{-1}]))$ and $J_s(L_K(E)) = M_2(J_s(K[x, x^{-1}]))$.

Corollary 4.8. *Let K be a any field, E a finite no-cycle graph and $\{v_1, ..., v_k\}$ the set of sinks. Then*

$$J(L_K(E)) = J_s(L_K(E)) = 0.$$

Proof. It immediately follows from Theorem 4.6. □

Remark 4.9. We can use Theorem 3.3 to proof Corollary 4.8. Especially, from Theorem 3.3 we have

$$L_K(E) \cong \mathop{\oplus}_{i=1}^{t} M_{n_i}(K),$$

where $\{w_1, ..., w_t\}$ the set of sinks for each $1 \le i \le t$, n_i is the number of path ending in the sink w_i (this includes w_i itself, as a path of length 0).

Fom Theorem 2.4, we have

$$J(L_K(E)) = \mathop{\oplus}_{i=1}^{t} J(M_{n_i}(K)),\ J_s(L_K(E)) = \mathop{\oplus}_{i=1}^{t} J_s(M_{n_i}(K)).$$

Fom Theorem 2.3, we have

$$J(L_K(E)) = \mathop{\oplus}_{i=1}^{t} M_{n_i}(J(K)),\ J_s(L_K(E)) = \mathop{\oplus}_{i=1}^{t} M_{n_i}(J_s(K)).$$

From Corollary 2.2, $J(K) = J_s(K) = 0.$ We have $J(L_K(E)) = J_s(L_K(E)) = 0.$

Example 4.10. (a) Let K be a field and E a graph has form Figure 5 or Figure 6. Since Figure 5 or Figure 6 graphs is acyclic, follow Corollary 4.8 $J(L_K(E)) = J_s(L_K(E)) = 0.$ This is also the result in Corollary 4.5 (c).

(b) Let K be is a field and E a acyclic graph has form in Example 3.4. From Corollary 4.8,

$$J(L_K(E)) = J_s(L_K(E)) = 0.$$

5. Conclusion

We have calculated the $J-$radical and the J_s-radical for the Leavitt path algebras $L_R(E)$ with coefficients in a commutative semiring R of some finite graphs E (Proposition 4.1, Proposition 4.2, Proposition 4.3, Proposition 4.4). In particular, we have also calculated the $J-$radical and the J_s-radical for the Leavitt path algebras $L_K(E)$ with coefficients in a field K of acyclic graphs (Corollary 4.8), no-exit graphs (Theorem 4.6) and applicable examples (Example 4.7 and Example 4.10).

In the future, we will expand two structural theorems (Theorem 3.3 and Theorem 3.5) of the Leavitt path algebras over commutative semirings of acyclic graphs and no-exit graphs.

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