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SUFFICIENT CONDITION FOR IDENTICAL SYNCHRONIZATION IN COMPLETE NETWORK OF ORDINARY DIFFERENTIAL EQUATIONS OF THE HINDMARSH – ROSE 3D TYPE WITH LINEAR COUPLING

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Abstract

This paper examines identical synchronization in a complete network consisting of n nodes. Each node is connected to every other node through linear coupling and is represented by ordinary differential equations of the Hindmarsh-Rose 3D type, which can be derived from the well-known Hodgkin-Huxley model. The study establishes a sufficient condition regarding the coupling strength necessary to achieve the desired synchronization. The findings indicate that networks with higher in-degrees for the nodes synchronize more readily. Additionally, the paper presents numerical simulations in C++ to support this theoretical result, highlighting the existence of a trade-off.

Keywords: *complete network, coupling strength, Hindmarsh-Rose 3D type, identical synchronization, linear coupling, ordinary differential equations.*

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ĐIỀU KIỆN ĐỦ CHO SỰ CỘNG HƯỞNG ĐỒNG NHẤT TRONG MẠNG LƯỚI ĐẦY ĐỦ CÁC PHƯƠNG TRÌNH VI PHÂN DẠNG HINDMARSH – ROSE 3D VỚI LIÊN KẾT TUYẾN TÍNH

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Tóm tắt

Bài viết này tập trung phân tích hiện tượng cộng hưởng đồng nhất trong một mạng lưới đầy đủ gồm n nút, trong đó mỗi nút được kết nối tuyến tính với tất cả các nút còn lại. Mỗi nút trong mạng được mô hình hóa bằng hệ phương trình vi phân ba chiều dạng Hindmarsh-Rose – một phiên bản đơn giản hóa của mô hình thần kinh nổi tiếng Hodgkin-Huxley. Dựa trên cấu trúc mạng đầy đủ, nghiên cứu xác định điều kiện đủ về độ mạnh liên kết để đảm bảo sự xuất hiện của cộng hưởng đồng nhất. Kết quả chỉ ra rằng số lượng liên kết đầu vào vào một nút tăng lên, khả năng xảy ra cộng hưởng cũng dễ dàng hơn. Bên cạnh đó, nghiên cứu còn tiến hành kiểm chứng lý thuyết thông qua mô phỏng số bằng ngôn ngữ lập trình C++ và xem xét mức độ tương quan giữa hai phương pháp.

Từ khóa: *độ mạnh liên kết, hệ phương trình vi phân, liên kết tuyến tính, mạng lưới đầy đủ, mô hình Hindmarsh-Rose 3D, sự cộng hưởng đồng nhất.*

1. Introduction

Synchronization is a common phenomenon that has been studied in various natural systems and in the field of nonlinear science. The term "synchronization" originates from Greek, where "syn" means "common" and "chronous" means "time". It refers to the occurrence of two systems exhibiting the same behavior simultaneously (Aziz-Alaoui, 2006). When two dynamic systems are synchronized, one system mimics the movement of the other. When many systems exhibit synchronized behavior, they are collectively referred to as synchronous systems. Research by Aziz-Alaoui (2006) and Corson (2009) suggests that synchronization can occur in networks of weakly coupled oscillators. Numerous applications have emerged from the study of synchronization, including enhancing laser power, controlling oscillations in chemical reactions, encoding electronic messages for secure communications, and synchronizing the output of electr1ical circuits (Ambrosio et al., 2012; Aziz-Alaoui, 2006).

Synchronization has been extensively studied across various fields, as it is a phenomenon observed in many natural occurrences. Examples include the movement of birds forming flocks, schools of fish in a lake, parades, and the reception and transmission of signals among groups of cells (Ambrosio et al., 2012; Ambrosio et al., 2013; Corson, 2009; Ermentrout et al., 2009; Hodgkin et al., 1952; Izhikievich, 2007; Keener et al., 2009). Therefore, studying synchronization is essential. This paper particularly focuses on cellular networks.

In the human brain, numerous cells connect to form intricate networks. A cellular network is defined as a system of cells that are physiologically interconnected. Their interactions primarily rely on electrochemical processes. This study examines the conditions necessary for achieving synchronization within a complete network of cells. Specifically, each cell is modeled using a system of ordinary differential equations of the Hindmarsh-Rose 3D type. To simplify the analysis, we investigate a complete network of n neurons that are interconnected using linear coupling.

In 1952, Hodgkin and Huxley introduced a four-dimensional mathematical model that approximated the electrical properties of cell voltage (Ambrosio et al., 2013; Corson, 2009; Izhikevich, 2007). Building on this model, many simpler models have been developed to describe cell voltage dynamics. In 1984, Hindmarsh and Rose published the Hindmarsh-Rose model (HR), a simplified three-dimensional model derived from Hodgkin and Huxley's renowned system of equations (Hodgkin et al., 1952). Despite its simplicity, the Hindmarsh-Rose model offers many notable analytical results and retains important biological significance. It effectively represents the equilibrium, activity, and burst behavior of cell voltage. The ordinary differential equations of the Hindmarsh-Rose 3D type (HR) are given by:

$$\begin{cases} \dfrac{dx}{dt} = -x^3 + ax^2 + y - z + I, \\ \dfrac{dy}{dt} = 1 - bx^2 - y, \\ \dfrac{dz}{dt} = r(s(x-c) - z), \end{cases} \tag{1}$$

where x represents the membrane voltage, y represents the fast-moving ionic currents across the cell membrane, and z represents the slow-moving ionic currents across the cell membrane. The parameters a,b,c,r,s $(a,b,r,s > 0)$are constants determined by practical experience, I presents the external current.

System (1) is considered as a neural model and from this, a network of n coupled systems (1) based on HR type is constructed as follows:

$$\begin{cases} \dfrac{dx_i}{dt} = -x_i^3 + ax_i^2 + y_i - z_i + I - h(x_i, x_j), \\ \dfrac{dy_i}{dt} = 1 - bx_i^2 - y_i, & i, j = 1, 2, ..., n, i \neq j, \\ \dfrac{dz_i}{dt} = r(s(x_i - c) - z_i), \end{cases} \tag{2}$$

where $x_i, y_i, z_i, i = 1, 2$ are the variables of the ith cell.

Function h is the coupling function that determines the type of connection between neurons ith and jth. Connections between neurons are essential and can be categorized into two types: chemical connections and electrical connections. Chemical connections are more abundant than electrical connections. For the purposes of this research, this paper will focus solely on electrical connections, where the coupling function is assumed to be linear (Ambrosio et al., 2013; Corson, 2009; Ermentrout et al., 2009; Izhikenvich, 2007) and is given by the following formula: $h(x_i, x_j) = g_{syn} \sum\limits_{j=1, j\neq i}^{n} c_{ij}(x_i - x_j), \quad i = 1, 2, ..., n.$

The parameter g_{syn} represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n\times n}$, defined by: $c_{\text{ij}} = 1$ if neurons ith and jth are coupled, $c_{\text{ij}} = 0$ if neurons ith and jth are not coupled, where $i, j = 1, 2, ..., n, i \neq j.$

In recent years, there has been a growing body of research on the synchronization of networks of cells. However, most of this research focuses only on cells stimulated by ordinary differential equations of the FitzHugh-Nagumo type (Ambrosio et al., 2012; Ambrosio et al., 2013) or includes a limited number of numerical studies on the ordinary differential equations of the Hindmarsh-Rose 2D (and even 3D) type (Corson, 2009). Notably, there has been a lack of analytical research concerning ordinary differential equations of the Hindmarsh-Rose 3D type within a complete network of cells. This gap highlights the significance of further exploration in this area, which offers valuable practical applications for contemporary applied mathematics.

2. Sufficient condition for identical synchronization in complete network of ordinary differential equations of the Hindmarsh-Rose 3D type with linear coupling

This paper investigates synchronization within a complete network, where each node is connected to all other nodes (Corson, 2009). For example, Figure 1 illustrates complete graphs ranging from 3 to 10 nodes. In this context, each node represents a neuron, which is modeled using a system of ordinary differential equations of the Hindmarsh-Rose 3D type. Each edge in the network signifies a synaptic connection, modeled by a linear coupling function. A network of n neurons that are bi-directionally coupled through electrical synapses based on the Hindmarsh-Rose model is described as follows:

$$\begin{cases} \dfrac{dx_i}{dt} = -x_i^3 + ax_i^2 + y_i - z_i + I - g_{syn} \sum\limits_{j=1, j\neq i}^{n} (x_i - x_j), \\ \dfrac{dy_i}{dt} = 1 - bx_i^2 - y_i, & i = 1, 2, ..., n, \\ \dfrac{dz_i}{dt} = r(s(x_i - c) - z_i), \end{cases} \tag{3}$$

where g_{syn} is the coupling strength between neurons ith and jth.

Definition 1 (Ambrosio et al., 2012).

Let $S_i = (x_i, y_i, z_i),\ i = 1,2,...,n$ *and* $S = (S_1, S_2, ..., S_n)$ *be a networki. We say that* S *is identically synchronous if we have:*

$$\lim_{t\to\infty}\left|x_i(t) - x_j(t)\right| = 0,\ \lim_{t\to\infty}\left|y_i(t) - y_j(t)\right| = 0 \text{ and}$$

$$\lim_{t\to\infty}\left|z_i(t) - z_j(t)\right| = 0,\ i, j = 1,2,...,n, i \neq j,$$

for every initial conditions $x_i(0), y_i(0), z_i(0), i = 1,2,...,n.$

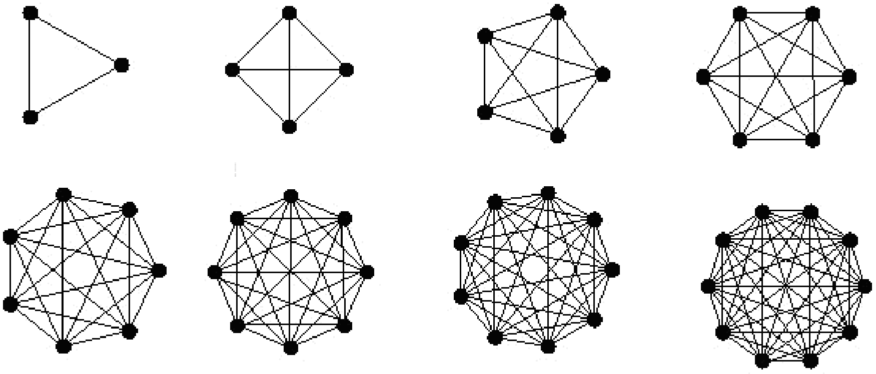


Figure 1. Complete graphs from 3 to 10 nodes.

System (3) can be rewritten as follows:

$$\begin{cases}
\dfrac{dx_i}{dt} = -x_i^3 + ax_i^2 + y_i - z_i + I - g_{syn}\displaystyle\sum_{j=1, j\neq i}^{n}(x_i - x_j), \\
\dfrac{dy_i}{dt} = 1 - bx_i^2 - y_i, \\
\dfrac{dz_i}{dt} = r(s(x_i - c) - z_i), & i = 2,...,n. \\
\dfrac{dx_1}{dt} = -x_1^3 + ax_1^2 + y_1 - z_1 + I - g_{syn}\displaystyle\sum_{j=2}^{n}(x_1 - x_j), \\
\dfrac{dy_1}{dt} = 1 - bx_1^2 - y_1, \\
\dfrac{dz_1}{dt} = r(s(x_1 - c) - z_1),
\end{cases} \tag{4}$$

Let $X = x_i - x_1, Y = y_i - y_1, Z = z_i - z_1$ and $U = z_i + z_1, i = 2,...,n.$ We have then the system corresponding to the variables X, Y, Z as follows:

$$\begin{cases}
\dfrac{dX}{dt} = Y - Z - \dfrac{1}{4}X^3 + X(aU - \dfrac{3}{4}U^2 - ng_{syn}), \\
\dfrac{dY}{dt} = -bXU - Y, \\
\dfrac{dZ}{dt} = rsX - rZ,
\end{cases} \tag{5}$$

Theorem 1.

If the coupling strength g_{syn} *verifies the condition* $g_{syn} \geq \max\left\{\frac{a^2}{3n}, \frac{1}{4n\gamma} + \frac{(b-2a)^2}{4n(3-\gamma b^2)}\right\}$, *with* $0 < \gamma < \frac{3}{b^2}$, *for all initial conditions* $x_i(0), y_i(0), z_i(0), i = 1,2,...,n,$ *the system (4) will synchronize.*

Proof. Let's choose the Lyapunov function as follows:

$$E(X,Y,Z) = \frac{1}{2}X^2 + \frac{\gamma}{2}Y^2 + \frac{1}{2rs}Z^2,$$

where γ is a positive constant. By taking derivative of this Lyapunov function according to t, we have:

$$\frac{dE(X,Y,Z)}{dt} = -\frac{X^4}{4} - \frac{Z^2}{s} - \left(AX^2 + BXY + \gamma Y^2\right),$$

where $A = \frac{3}{4}U^2 - aU + ng_{syn}, B = \gamma bU - 1.$

It can be seeen that $AX^2 + BXY + \gamma Y^2 > 0$ if the following two conditions are verified:

(i) Since $A = \frac{3}{4}U^2 - aU + ng_{syn}$ the solutions of the equation $A = 0$ are $U_{1,2} = \frac{2\left(a \pm \sqrt{a^2 - 3ng_{syn}}\right)}{3}$ if $g_{syn} \leq \frac{a^2}{3n}$. Therefore, $A > 0$ if $g_{syn} > \frac{a^2}{3n}$;

(ii) $\gamma A - \frac{B^2}{4} > 0 \Leftrightarrow (3-\gamma b^2)U^2 - 2(a-2b)U + 4ng_{syn} - \frac{1}{\gamma} > 0.$ This condition is satisfied if $g_{syn} > \frac{1}{4n\gamma} + \frac{(b-2a)^2}{4n(3-\gamma b^2)}$ and $\gamma < \frac{3}{b^2}$.

Then, if the coupling strength g_{syn} verifies the condition:

$$g_{syn} \geq \max\left\{\frac{a^2}{3n}, \frac{1}{4n\gamma} + \frac{(b-2a)^2}{4n(3-\gamma b^2)}\right\} \text{ with } 0 < \gamma < \frac{3}{b^2},$$

we have $AX^2 + BXY + \gamma Y^2 > 0.$

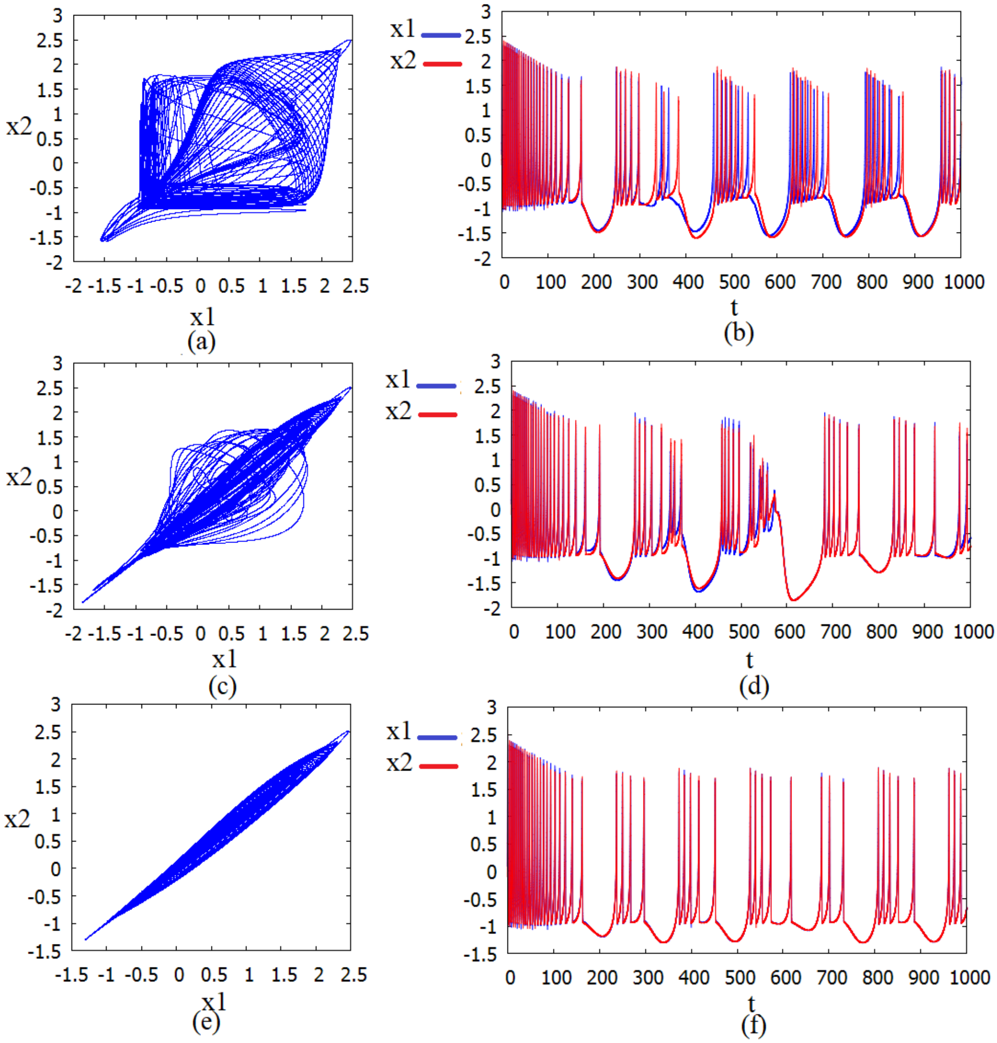
It leads to $\frac{dE(X,Y,Z)}{dt} < 0,$ for all X, Y, Z. It implies that the origin is globally asymptotically stable for $E(X,Y)$ (see Aeyels, 1995). Hence, the neurons of the network (4) is globally asymptotically synchronized. The theorem has been proven. ■

Remark 1. The above theorem offers a sufficient condition for the synchronization of system (4), but it is not a necessary condition. This means that even if the coupling strength does not meet the criteria outlined in Theorem 1, it does not imply that system (4) cannot synchronize. In fact, numerical methods confirm that there are instances where the coupling strength does not satisfy the sufficient condition, yet system (4) remains synchronous, as illustrated in Figures 2(e) and 2(f).

3. Numerical results and discussion

To check the effectiveness of the above-mentioned sufficient condition, the numerical results of the paper are performed in C++ and the Runge-Kutta algorithm is used to integrate the system (4) for $n=2$ with the time step $\Delta t = 0.001$. The parameter values are selected as follows: $a=3, b=5, c=-1.56, r=0.006, s=4, I=3$. with the initial conditions: $\left(x_1(0), y_1(0), z_1(0), x_2(0), y_2(0), z_2(0)\right) = \left(0.1, 0, 0.1, -0.1, 0.1, 0\right)$.

By numerical method, we get the results as shown in Figure 2.



(a) (b)

(c) (d)

(e) (f)

Figure 2. Before synchronization of the system (4) with $g_{syn}=0.1$ and $g_{syn}=0.4$ according to the figure (a), (b), (c), (d) since there is no identity between x_1 and x_2. In figure (e), with $g_{syn}=0.5$ we see that there is appearance of a diagonal in the plane Ox_1x_2, it means that there is an identity between x_1 and x_2. In figure (f), the evolution of x_1 in time is blue and the evolution of x_2 in time is red, in this figure we also see the identity between x_1 and x_2. Therefore, synchronization occurs in the network (4) for $n=2$.

The results show the correlation between the variables x_1 and x_2. If there is an identity between x_1 and x_2 then we say the system (4) for $n=2$ has synchronization, the case otherwise no synchronization occurs. Specifically:

- with $g_{syn} = 0.1$ then the sufficient condition is not satisfied (see Figure 2(a) and 2(b)). Synchronization does not occur in the network (4).

- with $g_{syn} = 0.4$ then the sufficient condition is not satisfied (see Figure 2(c) and 2(d)). Synchronization does not occur in the network (4).

- with $g_{syn} = 0.5$ then the sufficient condition is not satisfied (see Figure 2(e) and 2(f))., we see that there is appearance of a diagonal which means there is an identity between x_1 and x_2. In Figure 2(f), the evolution of x_1 in time is blue and the evolution of x_2 in time is red, in this figure we also see the identity between x_1 and x_2. Therefore, synchronization occurs in the network (4).

From the above result, in the case of two linearly coupled neurons, the coupling strength over or equal to $g_{syn} = 0.5$, these neurons has synchronous behaviors. By doing similarly for the complete networks of linearly identical coupled neurons, the values of coupling strength according to the number of neurons *n* are reported in Table 1. In Table 1, for each value of *n*, we seek one necessary value of coupling strength to get the synchronization in complete network corresponding to *n* from 2 to 20.

Table 1. Minimal coupling strength necessary to observe the synchronization

n		2	3	4	5
g_{syn}		0.5	0.35	0.31	0.28
n	6	7	8	9	10
g_{syn}	0.26	0.25	0.245	0.24	0.235
n	11	12	13	14	15
g_{syn}	0.23	0.228	0.225	0.223	0.222
n	16	17	18	19	20
g_{syn}	0.22	0.219	0.217	0.216	0.215

After conducting these numerical experiments, it becomes evident that the coupling strength necessary for observing synchronization among n neurons varies depending on the number of neurons present. The points depicted in Figure 3 illustrate the relationship between the coupling strength required for synchronization and the number of neurons in a complete network, as detailed in Table 1. We can derive a function that represents this relationship between the number of neurons n and the coupling strength listed in Table 1. The function is as follows:

$$g_{syn} = \frac{0.3}{n-1} + 0.2, \tag{6}$$

In Figure 3, the function described in equation (6) is represented by a curve, where the points corresponding to the coupling strengths are nearly aligned. This indicates that the coupling strength required for synchronization in a complete network follows the law outlined in (6). The simulations demonstrate that as the number of neurons increases, the necessary coupling strength decreases. In other words, achieving synchronization becomes easier as the number of neurons in complete networks grows.

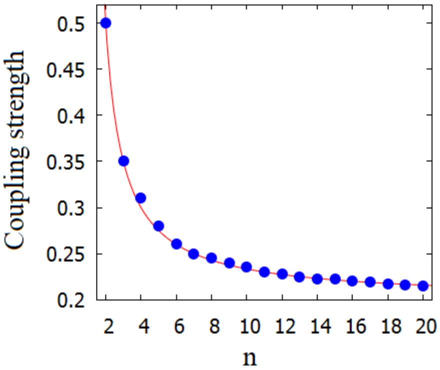


Figure 3. The evolution of the coupling strength with respect to the number of neurons

4. Conclusion

This study presents a sufficient condition for achieving synchronization in a complete network of n linearly coupled ordinary differential equations of the Hindmarsh-Rose 3D type. Theorem 1 indicates that as the value of n increases, the required coupling strength g_{syn} decreases. Numerical analysis demonstrates that synchronization remains stable when the coupling strength exceeds a certain threshold, which is influenced by the number of neurons in the network. Specifically, a larger number of neurons facilitates the phenomenon of synchronization. Therefore, a compromise can be reached between the theoretical and numerical findings. Additionally, further research is needed to explore different synchronization regimes in free networks coupled with chemical synapses.

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