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CONVERGENCE OF THREE-STEP ITERATION FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

Nguyen Trung Hieu[1] and Le Thi Phuong[2*]

[1]*Faculty of Mathematics - Informatics Teacher Education, School of Education, Dong Thap University, Cao Lanh 870000, Vietnam*

[2]*Student, Faculty of Mathematics - Informatics Teacher Education, School of Education, Dong Thap University, Cao Lanh 870000, Vietnam*

**Corresponding author, Email: 0022410312@student.dthu.edu.vn*

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Abstract

In this paper, we introduce a three-step iteration scheme for approximating common fixed points of three asymptotically nonexpansive mappings in $CAT(0)$ spaces. We also establish and prove some Δ-convergence and strong convergence results of this iterative sequence to a common fixed points of three asymptotically nonexpansive mappings in complete $CAT(0)$ spaces. In addition, we give an example to illustrate the convergence results obtained.

Keywords: *asymptotically nonexpansive mapping, $CAT(0)$ space, common fixed point, three-step iteration scheme.*

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NGHIÊN CỨU SỰ HỘI TỤ CỦA DÃY LẶP BA BƯỚC CHO ÁNH XẠ KHÔNG GIÃN TIỆM CẬN TRONG KHÔNG GIAN CAT(0)

Nguyễn Trung Hiếu[1] và Lê Thị Phương[2*]

[1]Khoa Sư phạm Toán - Tin, Trường Sư phạm, Trường Đại học Đồng Tháp, Việt Nam

[2]Sinh viên, Khoa Sư phạm Toán - Tin, Trường Sư phạm, Trường Đại học Đồng Tháp, Việt Nam

**Tác giả liên hệ, Email: 0022410312@student.dthu.edu.vn*

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Tóm tắt

Trong bài báo này, chúng tôi giới thiệu một dãy lặp ba bước để xấp xỉ điểm bất động chung của ba ánh xạ không giãn tiệm cận trong không gian $CAT(0)$*. Sau đó, chúng tôi thiết lập và chứng minh một số kết quả về* Δ*- hội tụ và hội tụ mạnh của dãy lặp này đến điểm bất động chung của ba ánh xạ không giãn tiệm cận trong không gian* $CAT(0)$ *đầy đủ. Đồng thời, chúng tôi đưa ra ví dụ minh họa cho kết quả hội tụ đạt được.*

Từ khóa: *ánh xạ không giãn tiệm cận, dãy lặp ba bước, điểm bất động chung, không gian* $CAT(0)$.

1. Introduction

In fixed point theory, the study of approximating fixed points of mappings by different iteration processes has been conducted (Cegielski, 2012; Sabri, 2025; Tufa, 2022). In addition, some authors have focused on establishing convergence results of iteration processes to fixed points of mappings in generalized spaces. In recent years, one of the generalized spaces that has attracted the attention of many authors is the $CAT(0)$ space. A metric space X is a $CAT(0)$ space if X is geodesically connected and every geodesic triangle in X is "thinner" or at least not "thicker" than its comparison triangle in the Euclidean plane (Bridson & Haefliger, 1999). A $CAT(0)$ space is a generalization of a Hadamard manifold, which is a complete, simply connected Riemannian manifold whose sectional curvature is non-positive; in addition, every complete and simply connected Riemannian manifold with non-positive sectional curvature is a $CAT(0)$ space. Some other examples of $CAT(0)$ spaces include the complex Hilbert ball endowed with the hyperbolic metric, pre-Hilbert spaces…A $CAT(0)$ space plays an important role in many areas of mathematics and has numerous applications in computer science (Bridson & Haefliger, 1999).

Fixed point theory in $CAT(0)$ spaces was first studied by Kirk (Kirk, 2003; Kirk, 2004) in 2003. Kirk showed that every nonexpansive mapping defined on a bounded closed convex subset of a complete $CAT(0)$ space has a fixed point. After that, fixed point theory for the classes of nonexpansive mappings and generalized nonexpansive mappings in a $CAT(0)$ space continues to be studied. One of the research lines on this domain is the generalization of existing iterative processes to construct new iterative processes and the application of these iterative processes to study the convergence to fixed points of nonexpansive mappings and generalized nonexpansive mappings in a $CAT(0)$ space (Dhompongsa & Panyanak, 2008; Khan & Abbas, 2011; Tufa, 2023).

For the class of asymptotically nonexpansive mappings, some convergence results of various iteration processes to a fixed point of each mapping in $CAT(0)$ space have been established. Nanjaras and Panyanak (2010) proved the demiclosed principle for asymptotically nonexpansive mappings in $CAT(0)$ space; based on this, the authors have proved results on the Δ-convergence of the Krasnosel'skii-Mann iteration to a fixed point of an asymptotically nonexpansive mapping in $CAT(0)$ space:

$$x_{n+1} = \alpha_n T^n x_n \oplus (1-\alpha_n)x_n,$$

where $\{\alpha_n\} \subset [0,1]$. Meanwhile, Niwongsa and Panyanak (2010) proved some results on Δ-convergence and strong convergence of the Noor iteration for an asymptotically nonexpansive mapping in $CAT(0)$ space:

$$\begin{cases} z_n = \gamma_n T^n x_n \oplus (1-\gamma_n)x_n \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n)T^n x_n, \end{cases}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset [0,1]$. In 2013, Sahin and Basarir (2013), proved some results on strong convergence of the modified S-iteration process for an asymptotically nonexpansive mapping in $CAT(0)$ space:

$$\begin{cases} y_n = \beta_n T^n x_n \oplus (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n) T^n x_n, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. In 2021, Yambangwai and Thianwan (2021) proved the convergence of the following three-step iteration to a fixed point of an asymptotically nonexpansive mapping in $CAT(0)$ space:

$$\begin{cases} z_n = \gamma_n T^n x_n \oplus (1-\gamma_n) x_n, \\ y_n = \beta_n T^n z_n \oplus (1-\beta_n) z_n, \\ x_{n+1} = \alpha_n T^n y_n \oplus (1-\alpha_n) T^n z_n, \end{cases} \tag{1.1}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset [0,1]$. Note that most previous works mainly considered iteration processes for an asymptotically nonexpansive mapping, while the case of three asymptotically nonexpansive mappings in $CAT(0)$ spaces has not yet been studied. Therefore, an interesting work naturally raised is to construct a three-step iterative process for three asymptotically nonexpansive mappings to study its convergence to a common fixed point of these asymptotically nonexpansive mappings in $CAT(0)$ spaces. Motivated by the iterative process (1.1), we construct the following three-step iterative process to study the convergence to a common fixed point of three asymptotically nonexpansive mappings in $CAT(0)$ spaces:

$$\begin{cases} z_n = \gamma_n T_3^n x_n \oplus (1-\gamma_n) x_n, \\ y_n = \beta_n T_2^n z_n \oplus (1-\beta_n) z_n, \\ x_{n+1} = \alpha_n T_1^n y_n \oplus (1-\alpha_n) T_2^n z_n, \end{cases} \tag{1.2}$$

where $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset [0,1]$. Then, by using the geometric properties of $CAT(0)$ spaces and the fundamental tools of fixed point theory in $CAT(0)$ spaces, we establish and prove some convergence results of the iterative process (1.2) to a common fixed point of three asymptotically nonexpansive mappings in $CAT(0)$ spaces. Furthermore, we provide an illustrative example for the convergence of this iterative process and show that the iterative process (1.2) converges faster than the iterative process in (Sabri, 2025). The obtained results extend the main results in (Yambangwai & Thianwan, 2021) from a three-step iterative process for one asymptotically nonexpansive mapping to a three-step iterative process for three asymptotically nonexpansive mappings in $CAT(0)$ spaces.

2. Preliminaries

In this section, we will present some basic definitions and results used throughout the paper.

Definition 2.1. Let X be a nonempty set and a mapping $d : X \times X \to \mathbb{R}$ the following three conditions for all $x, y, z \in X$.

(1) $d(x, y) \geq 0,\ d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$.

Then d is said to be a *metric* in X , (X, d) is said to be a *metric space* and $d(x, y)$ is said to be *the distance between two points* x *and* y.

Let (X,d) be a metric space and $x,y\in X$. A *geodesic path* joining u to v is a map $c:[0,l]\subset\mathbb{R}\to X$ such that $c(0)=x, c(l)=y$ and $d(c(t),c(t'))=|t-t'|$ for all $t,t'\in[0,l]$. The image of c is said to be a *geodesic segment* joining x and y, is denoted by $[x,\ y]$.

The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic. The space (X,d) is said to be *uniquely geodesic* if for each $x,y\in X$ there is exactly one geodesic joining x and y.

A *geodesic triangle* $\Delta(x_1,x_2,x_3)$ in a geodesic space (X,d) consists of three points x_1,x_2,x_3 in X (the vertices of Δ) and every pair of points is joined by a geodesic segment (the edges of Δ). A *comparison triangle* for the geodesic triangle $\Delta(x_1,x_2,x_3)$ in (X,d) is a triangle $\bar{\Delta}(x_1,x_2,x_3):=\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that

$$d_{\mathbb{R}^2}(\bar{x}_i,\bar{x}_j)=d(x_i,x_j) \text{ for } i,j\in\{1,2,3\}.$$

Definition 2.2. (Chang et al., 2012) A geodesic space (X,d) is said to be a $CAT(0)$ *space* if for every geodesic triangle $\Delta(x_1,x_2,x_3)$ in X and its comparison triangle $\bar{\Delta}:=\Delta(\bar{x}_1,\bar{x}_2,\bar{x}_3)$ in $\mathbb{R}^2$, the following inequality is satisfied

$$d(x,y)\le d_{\mathbb{R}^2}(\bar{x},\bar{y})$$

for all $x,y\in\Delta$ and $\bar{x},\bar{y}\in\bar{\Delta}$.

Proposition 2.3. (Dhompongsa & Panyanak, 2008, Lemma 2.1, Lemma 2.4, Lemma 2.5) *Let* (X,d) *be a* $CAT(0)$ *space. Then*

(i) For $x,y\in X$ *and* $t\in[0,1]$, *there exists a unique point* $z\in[x,y]$ *such that*

$$d(x,z)=td(x,y) \text{ and } \mathrm{d}(y,z)=(1-t)d(x,y).. \tag{2.1}$$

We denote by $(1-t)x\oplus ty$ *the point* z *in (2.1).*

(ii)For $x,y,z\in X$ *and* $t\in[0,1]$, *we have* $d((1-t)x\oplus ty,z)\le(1-t)d(x,z)+td(y,z)$.

(iii) For $x,y,z\in X$ *and* $t\in[0,1]$, *we have*

$$d^2((1-t)x\oplus ty,z)\le(1-t)d^2(x,z)+td^2(y,z)-t(1-t)d^2(x,y).$$

Let (X,d) be a metric space and $\{x_n\}$ be a bounded sequence in (X,d). For $x\in X$, we set $r(x,\{x_n\})=\limsup_{n\to\infty}d(x,x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\})=\inf\{r(x,\{x_n\}):x\in X\}$ and the *asymptotic center* $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\})=\{x\in X:r(x,\{x_n\})=r(\{x_n\})\}$.

In a complete $CAT(0)$ space, $A(\{x_n\})$ consists of exactly one point.

Definition 2.4. (Kirk & Panyanak, 2008; Lim, 1976) Let (X,d) be a metric space, $\{x_n\}$ is said to Δ - *converge* to $x\in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n=x$ and call x the Δ -*limit* of $\{x_n\}$.

Proposition 2.5. (Dhompongsa et al., 2007, Proposition 2.1) *Let (X,d) be a complete $CAT(0)$ space and C be a closed convex subset of X. If $\{x_n\}$ is a bounded sequence in C, then the asymptotic center of $\{x_n\}$ in C exists and is unique.*

Proposition 2.6. (Dhompongsa & Panyanak, 2008, Lemma 2.8) *Let (X,d) be a complete $CAT(0)$ space, $\{x_n\}$ be a bounded sequence in X, $A(\{x_n\})=\{x\}$ and $\{u_n\}$ be a subsequence of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$ and $\{d(x_n,u)\}$ converge. Then $x=u$.*

Proposition 2.7. (Kirk & Panyanak, 2008, Proposition 3.5) *Let (X,d) be a complete $CAT(0)$ space and $\{x_n\}$ is a bounded sequence in X. Then, $\{x_n\}$ has a subsequence Δ -convergent.*

Proposition 2.8. (Zhou et al., 2002, Lemma 1.2) *Let a_n and b_n be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \le (1+b_n)a_n, n \ge 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Next, we will present the notion of asymptotically nonexpansive mappings and some results related to the fixed points of this class of mappings in complete $CAT(0)$ spaces.

Definition 2.9. (Yambangwai & Thianwan, 2021, p.816) Let (X,d) be a metric space and the mapping $T: X \to X$. Then

(1) T is said to be *nonexpansive* if $d(Tx,Ty) \le d(x,y)$ for all $x,y \in X$.

(2) T is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that for all $x,y \in X$, we have $d(T^n x, T^n y) \le k_n d(x,y)$ with $n \ge 1$.

The following example demonstrates the definition of an asymptotically nonexpansive mapping.

Example 2.10. (Yambangwai & Thianwan, 2021, p.822) Let $X=\mathbb{R}$ be a metric space with $d(x,y)=|x-y|$, $x,y \in \mathbb{R}$ and $C=[-1,1]$. Consider four mappings $T_1,T_2,T_3,T_4 : C \to C$ defined by:

$$T_1 x = \begin{cases} -2\sin\dfrac{x}{2}, & x \in [0,1], \\ 2\sin\dfrac{x}{2}, & \mathrm{x} \in [\text{-}1,0) \end{cases}, \quad T_2 x = \begin{cases} x, & x \in [0,1], \\ -x, & x \in [-1,0) \end{cases} \quad T_3 x = \sin x \text{ and } T_4 x = \frac{x}{2}.$$

Then, T_1,T_2,T_3,T_4 are asymptotically nonexpansive mappings with $k_n = 1, n \in \mathbb{N}^*$.

Theorem 2.11. (Yambangwai & Thianwan, 2021, Theorem 2.1) *Let (X,d) be a complete $CAT(0)$ space, C be a nonempty bounded closed and convex subset of X and $T: C \to C$ be asymptotically nonexpansive. Then T has a fixed point.*

Proposition 2.12. (Yambangwai et al., 2020, Theorem 12) *Let (X,d) be a complete $CAT(0)$ space, C be a closed and convex subset of X and $T: C \to C$ be asymptotically nonexpansive. Suppose that $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and Δ-$\lim_{n\to\infty} x_n$. Then $x = Tx$.*

Definition 2.13. (Sridarat et al., 2019) Let (X,d) be $CAT(0)$ space, C be a closed and convex subset of X. Then three mappings $T_1,T_2,T_3 : C \to C$ are said to satisfy the condition (C) if there exists a non-decreasing function $f:[0,\infty)\to[0,\infty)$ with $f(0)=0$ and $f(r)>0$ for all $r\in(0,\infty)$ such that

$$\max\left\{d(x,T_1x), d(x,T_2x), d(x,T_3x)\right\} \geq f(d(x,\mathcal{F}))$$

for all $x \in C$, where $d(x,\mathcal{F}) = \inf\{d(x,y) : y \in \mathcal{F}\}$ and $\mathcal{F} = F(T_1)\cap F(T_2)\cap F(T_3)$.

3. Main results

First, we establish some properties of the iterative process (1.2), where T_1, T_2 and T_3 are three asymptotically nonexpansive mappings in a $CAT(0)$ space.

Proposition 3.1. *Let (X,d) be a complete $CAT(0)$ space, C be a nonempty, closed, and convex subset of X, $T_1,T_2,T_3 : C \to C$ be three asymptotically nonexpansive mappings with $\{k_{i_n}\}$ satisfying $\{k_{i_n}\}\subset[1,\infty)$, $\sum_{n=1}^{\infty}(k_{i_n}-1)<\infty, i=1,2,3$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences of real numbers in $[0,1]$. For a given $x_1 \in C$, consider the sequence $\{x_n\}$ defined by (1.2). Then $\lim_{n\to\infty} d(x_n,x^*)$ exists for all $x^* \in \mathcal{F} = F(T_1)\cap F(T_2)\cap F(T_3)$.*

Proof. Put $k_{i_n} = 1+u_{i_n}$ with $n\geq 1, i=1,2,3.$ Since $\sum_{n=1}^{\infty}(k_{i_n}-1)<\infty$, we obtain $\sum_{n=1}^{\infty}u_{i_n}<\infty.$ Put $u_n = \max\left\{u_{i_n}, i=1,2,3\right\}$. For each $x^*\in\mathcal{F}$. Since T_3 is an asymptotically nonexpansive mapping and iterative process (1.2), we have

$$\begin{aligned} d(z_n,x^*) &= d(\gamma_n T_3^n x_n \oplus (1-\gamma_n)x_n, x^*) \\ &\leq (1-\gamma_n)d(x_n,x^*) + \gamma_n d(T_3^n x_n, x^*) \\ &\leq (1-\gamma_n)d(x_n,x^*) + \gamma_n(1+u_n)d(x_n,x^*) \\ &= (1+\gamma_n u_n)d(x_n,x^*) \\ &\leq (1+u_n)d(x_n,x^*). \end{aligned} \tag{3.1}$$

Similarly, since T_2 is an asymptotically nonexpansive mapping, we have

$$\begin{aligned} d(y_n,x^*) &= d(\beta_n T_2^n z_n \oplus (1-\beta_n)z_n, x^*) \\ &\leq (1-\beta_n)d(z_n,x^*) + \beta_n d(T_2^n z_n, x^*) \\ &\leq (1-\beta_n)d(z_n,x^*) + \beta_n(1+u_n)d(z_n,x^*) \\ &= (1+\beta_n u_n)d(z_n,x^*) \\ &\leq (1+u_n)d(z_n,x^*). \end{aligned} \tag{3.2}$$

Since T_1, T_2 are asymptotically nonexpansive mappings, iterative process (1.2), combined with (3.1) and (3.2), we obtain

$$
\begin{aligned}
d(x_{n+1},x^*) &= d(\alpha_n T_1^n y_n \oplus (1-\alpha_n)T_2^n z_n, x^*)\\
&\le (1-\alpha_n)d(T_2^n z_n,x^*)+\alpha_n d(T_1^n y_n,x^*)\\
&\le (1-\alpha_n)(1+u_n)d(z_n,x^*)+\alpha_n(1+u_n)d(y_n,x^*)\\
&\le (1-\alpha_n)(1+u_n)d(z_n,x^*)\ +\alpha_n(1+u_n)(1+u_n)d(z_n,x^*)\\
&= (1+u_n)d(z_n,x^*)-\alpha_n(1+u_n)d(z_n,x^*)\\
&\quad +\alpha_n u_n(1+u_n)d(z_n,x^*)+\alpha_n(1+u_n)d(z_n,x^*)\\
&= (1+u_n)d(z_n,x^*)+\alpha_n u_n(1+u_n)d(z_n,x^*)\\
&\le (1+u_n)d(z_n,x^*)+u_n(1+u_n)d(z_n,x^*)\\
&= (1+2u_n+u_n^2)d(z_n,x^*)\\
&\le (1+2u_n+u_n^2)(1+u_n)d(x_n,x^*)\\
&= (1+(3u_n+3u_n^2+u_n^3))d(x_n,x^*).
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} u_n < \infty$, we have $\lim_{n\to\infty} u_n = 0$. By using Proposition 2.8, it follows that $\lim_{n\to\infty} d(x_n,x^*)$ exists. This completes the proof □

Proposition 3.2. *Let (X,d) be a complete $CAT(0)$ space, C be a nonempty, closed, and convex subset of $X, T_1,T_2,T_3 : C\to C$ be three asymptotically nonexpansive mappings with $\{k_{i_n}\}$ satisfying $\{k_{i_n}\}\subset[1,\infty)$, $\sum_{n=1}^{\infty}(k_{i_n}-1)<\infty, i=1,2,3$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences of real numbers in $[0,1]$, satisfying*

$$0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1, 0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1, 0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1.$$

For a given $x_1\in C$, consider the sequence $\{x_n\}$ defined by (1.2). Then

$$\lim_{n\to\infty} d(x_n,T_1x_n)=\lim_{n\to\infty} d(x_n,T_2x_n)=\lim_{n\to\infty} d(x_n,T_3x_n)=0.$$

Proof. Put $k_{i_n}=1+u_{i_n}$ with $n\ge 1, i=1,2,3.$ Since $\sum_{n=1}^{\infty}(k_{i_n}-1)<\infty$, we obtain $\sum_{n=1}^{\infty}u_{i_n}<\infty.$ Put $u_n=\max\left\{u_{i_n}, i=1,2,3\right\}$.

Step 1: We prove that $\lim_{n\to\infty} d(x_n,T_1^n x_n)=\lim_{n\to\infty} d(x_n,T_2^n x_n)=\lim_{n\to\infty} d(x_n,T_3^n x_n)=0.$

Since T_3 is an asymptotically nonexpansive mapping and iterative process (1.2), we have

$$
\begin{aligned}
d^2(z_n,x^*) &= d^2(\gamma_n T_3^n x_n \oplus (1-\gamma_n)x_n, x^*)\\
&\le \gamma_n d^2(T_3^n x_n,x^*)+(1-\gamma_n)d^2(x_n,x^*)\ -\gamma_n(1-\gamma_n)d^2(T_3^n x_n,x_n)\\
&\le \gamma_n(1+u_n)^2 d^2(x_n,x^*)+(1-\gamma_n)d^2(x_n,x^*)
\end{aligned}
$$

$$-\gamma_n(1-\gamma_n)d^2(T_3^n x_n, x_n)$$
$$=(\gamma_n+2\gamma_n u_n+\gamma_n u_n^2)d^2(x_n,x^*)+(1-\gamma_n)d^2(x_n,x^*)$$
$$-\gamma_n(1-\gamma_n)d^2(T_3^n x_n, x_n)$$
$$\leq(1+2u_n+u_n^2)d^2(x_n,x^*)-\gamma_n(1-\gamma_n)d^2(T_3^n x_n,x_n)$$
$$=(1+u_n)^2 d^2(x_n,x^*)-\gamma_n(1-\gamma_n)d^2(T_3^n x_n,x_n). \tag{3.3}$$

Similarly, since T_2 is an asymptotically nonexpansive mapping, we have

$$d^2(y_n,x^*)=d^2(\beta_n T_2^n z_n\oplus(1-\beta_n)z_n,x^*)$$
$$\leq\beta_n d^2(T_2^n z_n,x^*)+(1-\beta_n)d^2(z_n,x^*)-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)$$
$$\leq\beta_n(1+u_n)^2 d^2(z_n,x^*)+(1-\beta_n)d^2(z_n,x^*)-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)$$
$$=(\beta_n+2\beta_n u_n+\beta_n u_n^2)d^2(z_n,x^*)+(1-\beta_n)d^2(z_n,x^*)$$
$$-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)$$
$$\leq(1+2u_n+u_n^2)d^2(z_n,x^*)-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)$$
$$=(1+u_n)^2 d^2(z_n,x^*)-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n). \tag{3.4}$$

Since T_1,T_2 are asymptotically nonexpansive mappings, iterative process (1.2), by using (3.1) and (3.2), we obtain

$$d^2(x_{n+1},x^*)=d^2(\alpha_n T_1^n y_n\oplus(1-\alpha_n)T_2^n z_n,x^*)$$
$$\leq(1-\alpha_n)d^2(T_2^n z_n,x^*)+\alpha_n d^2(T_1^n y_n,x^*)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n T_2^n z_n)$$
$$\leq(1-\alpha_n)(1+u_n)^2 d^2(z_n,x^*)+\alpha_n(1+u_n)^2 d^2(y_n,x^*)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n)$$
$$\leq(1-\alpha_n)(1+u_n)^2 d^2(z_n,x^*)+\alpha_n(1+u_n)^4 d^2(z_n,x^*)$$
$$-\alpha_n(1+u_n)^2\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n)$$
$$\leq(1-\alpha_n)(1+u_n)^4 d^2(z_n,x^*)+\alpha_n(1+u_n)^4 d^2(z_n,x^*)$$
$$-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n)$$
$$=(1+u_n)^4 d^2(z_n,x^*)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n)-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)$$
$$\leq(1+u_n)^4[(1+u_n)^2 d^2(x_n,x^*)-\gamma_n(1-\gamma_n)d^2(T_3^n x_n,x_n)]$$
$$-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n)$$
$$\leq(1+u_n)^6 d^2(x_n,x^*)-\gamma_n(1-\gamma_n)d^2(T_3^n x_n,x_n)$$
$$-\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n)-\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n). \tag{3.5}$$

Since C is a bounded set, there exists $B_M[x_0]=\{x\in X: d(x,x_0)\leq M\}$ such that $C\subset B_M[x_0]$ with $M>0$. From (3.5), we have

$$\begin{aligned}\alpha_n(1-\alpha_n)d^2(T_1^n y_n,T_2^n z_n) &\le (1+u_n)^6 d^2(x_n,x^*)-d^2(x_{n+1},x^*)\\ &=(1+6u_n+15u_n^2+20u_n^3+15u_n^4+6u_n^5+u_n^6)d^2(x_n,x^*)-d^2(x_{n+1},x^*)\\ &=d^2(x_n,x^*)-d^2(x_{n+1},x^*)\\ &\quad+(6u_n+15u_n^2+20u_n^3+15u_n^4+6u_n^5+u_n^6)d^2(x_n,x^*)\\ &\le d^2(x_n,x^*)-d^2(x_{n+1},x^*)\\ &\quad+M^2(6u_n+6u_n^2+6u_n^3+6u_n^4+6u_n^5+u_n^6). \qquad (3.6)\end{aligned}$$

$$\begin{aligned}\beta_n(1-\beta_n)d^2(T_2^n z_n,z_n) &\le (1+u_n)^6 d^2(x_n,x^*)-d^2(x_{n+1},x^*)\\ &\le d^2(x_n,x^*)-d^2(x_{n+1},x^*)+M^2(6u_n+6u_n^2+6u_n^3+6u_n^4+6u_n^5+u_n^6). \qquad (3.7)\end{aligned}$$

$$\begin{aligned}\gamma_n(1-\gamma_n)d^2(T_3^n x_n,x_n) &\le (1+u_n)^6 d^2(x_n,x^*)-d^2(x_{n+1},x^*)\\ &\le d^2(x_n,x^*)-d^2(x_{n+1},x^*)+M^2(6u_n+6u_n^2+6u_n^3+6u_n^4+6u_n^5+u_n^6). \qquad (3.8)\end{aligned}$$

Since $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$ there exists a real number $a>0$ such that $\alpha_n(1-\alpha_n)\ge a>0.$ Using (3.6) and $\sum_{n=1}^{\infty}u_n<\infty,$ we obtain

$$\lim_{n\to\infty}d(T_1^n y_n,T_2^n z_n)=0. \qquad (3.9)$$

By a similar argument as above, using $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$, (3.7) and $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$, (3.8) and combine with $\sum_{n=1}^{\infty}u_n<\infty$, we conclude that

$$\lim_{n\to\infty}d(T_2^n z_n,z_n)=0, \qquad (3.10)$$

$$\lim_{n\to\infty}d(T_3^n x_n,x_n)=0. \qquad (3.11)$$

Since $z_n=\gamma_n T_3^n x_n\oplus(1-\gamma_n)x_n$, we get

$$\begin{aligned}d(z_n,x_n)&=d(\gamma_n T_3^n x_n\oplus(1-\gamma_n)x_n,x_n)\\ &\le(1-\gamma_n)d(x_n,x_n)+\gamma_n d(T_3^n x_n,x_n)\\ &=\gamma_n d(T_3^n x_n,x_n). \qquad (3.12)\end{aligned}$$

Taking the limit in (3.12) as $n\to\infty$, using (3.11) and noting that $\liminf_{n\to\infty}\gamma_n>0$, we have

$$\lim_{n\to\infty}d\left(z_n,x_n\right)=0. \qquad (3.13)$$

Furthermore, we have

$$\begin{aligned}d(T_1^n y_n,x_n)&\le d(T_1^n y_n,T_2^n z_n)+d(T_2^n z_n,x_n)\\ &\le d(T_1^n y_n,T_2^n z_n)+d(T_2^n z_n,z_n)+d(z_n,x_n). \qquad (3.14)\end{aligned}$$

Thus, by (3.9), (3.10), (3.13) and (3.14), we conclude

$$\lim_{n\to\infty} d\left(T_1^n y_n, x_n\right) = 0. \tag{3.15}$$

Since $y_n = \beta_n T_2^n z_n \oplus \left(1-\beta_n\right) z_n$, we get

$$\begin{aligned} d(y_n, z_n) &= d(\beta_n T_2^n z_n \oplus \left(1-\beta_n\right) z_n, z_n) \\ &\le \left(1-\beta_n\right) d(z_n, z_n) + \beta_n d(T_2^n z_n, z_n) \\ &= \beta_n d(T_2^n z_n, z_n). \end{aligned} \tag{3.16}$$

By using (3.10), (3.16) and noting that $\liminf_{n\to\infty} \beta_n > 0$, we obtain

$$\lim_{n\to\infty} d\left(y_n, z_n\right) = 0. \tag{3.17}$$

Since T_2 is an asymptotically nonexpansive mapping, we have

$$\begin{aligned} d(x_n, T_2^n x_n) &\le d(T_2^n x_n, T_2^n z_n) + d(T_2^n z_n, x_n) \\ &\le (1+u_n) d(x_n, z_n) + d(T_2^n z_n, x_n) \\ &\le (1+u_n) d(x_n, z_n) + d(T_2^n z_n, z_n) + d(z_n, x_n) \\ &= (2+u_n) d(x_n, z_n) + d(T_2^n z_n, z_n). \end{aligned} \tag{3.18}$$

Then, by combining (3.10), (3.13) with (3.18) and using $\lim_{n\to\infty} u_n = 0$, we conclude

$$\lim_{n\to\infty} d(x_n, T_2{}^n x_n) = 0. \tag{3.19}$$

Since T_1 is an asymptotically nonexpansive mapping, we have

$$\begin{aligned} d(x_n, T_1^n x_n) &\le d(T_1^n x_n, T_1^n y_n) + d(T_1^n y_n, x_n) \\ &\le (1+u_n) d(x_n, y_n) + d(T_1^n y_n, x_n) \\ &\le (1+u_n)(d(x_n, z_n) + d(y_n, z_n)) + d(T_1^n y_n, x_n). \end{aligned} \tag{3.20}$$

It follows from (3.10), (3.13), (3.17), (3.19) and using $\lim_{n\to\infty} u_n = 0$ that

$$\lim_{n\to\infty} d\left(x_n, T_1{}^n x_n\right) = 0. \tag{3.21}$$

Step 2: We prove that $\lim_{n\to\infty} d(x_n, T_1 x_n) = \lim_{n\to\infty} d(x_n, T_2 x_n) = \lim_{n\to\infty} d(x_n, T_3 x_n) = 0.$

Since $x_{n+1} = \alpha_n T_1^n y_n \oplus (1-\alpha_n) T_2^n z_n$, we get

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n T_1^n y_n \oplus (1-\alpha_n) T_2^n z_n, x_n) \\ &\le \alpha_n d(T_1^n y_n, x_n) + (1-\alpha_n) d(T_2^n z_n, x_n) \\ &\le \alpha_n d(T_1^n y_n, x_n) + (1-\alpha_n)(d(T_2^n z_n, z_n) + d(z_n, x_n)). \end{aligned} \tag{3.22}$$

Thus, by (3.10), (3.13),(3.17), (3.22) and noting that $\liminf_{n\to\infty} \alpha_n > 0$, we have

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{3.23}$$

Since T_1 is an asymptotically nonexpansive mapping, we have

$$\begin{aligned} d(x_{n+1}, T_1^n x_{n+1}) &\le d(x_{n+1}, x_n) + d(T_1^n x_{n+1}, T_1^n x_n) + d(T_1^n x_n, x_n) \\ &\le d(x_{n+1}, x_n) + k_n d(x_{n+1}, x_n) + d(T_1^n x_n, x_n) \\ &\le (1+k_n) d(x_{n+1}, x_n) + d(T_1^n x_n, x_n). \end{aligned} \tag{3.24}$$

It follows from (3.21), (3.23) and (3.24) that

$$\lim_{n\to\infty} d\left(x_{n+1}, T_1^n x_{n+1}\right) = 0. \tag{3.25}$$

Furthermore, we get

$$\begin{aligned} d(x_{n+1}, T_1 x_{n+1}) &\le d(x_{n+1}, T_1^{n+1} x_{n+1}) + d(T_1^{n+1} x_{n+1}, T_1 x_{n+1}) \\ &\le d(x_{n+1}, T_1^{n+1} x_{n+1}) + k_1 d(T_1^n x_{n+1}, x_{n+1}). \end{aligned} \tag{3.26}$$

Taking the limit in (3.26) as $n \to \infty$ and using (3.21), (3.25), we find that

$$\lim_{n\to\infty} d(x_n, T_1 x_n) = 0.$$

Since T_2 is an asymptotically nonexpansive mapping, we have

$$\begin{aligned} d\left(x_{n+1}, T_2^n x_{n+1}\right) &\le d\left(x_{n+1}, x_n\right) + d\left(T_2^n x_{n+1}, T_2^n x_n\right) + d\left(T_2^n x_n, x_n\right) \\ &\le d\left(x_{n+1}, x_n\right) + k_n d\left(x_{n+1}, x_n\right) + d\left(T_2^n x_n, x_n\right) \\ &\le \left(1+k_n\right) d\left(x_{n+1}, x_n\right) + d\left(T_2^n x_n, x_n\right). \end{aligned} \tag{3.27}$$

It follows from (3.19), (3.24) and (3.27) that

$$\lim_{n\to\infty} d\left(x_{n+1}, T_2^n x_{n+1}\right) = 0. \tag{3.28}$$

Furthermore, we have

$$\begin{aligned} d\left(x_{n+1}, T_2 x_{n+1}\right) &\le d\left(x_{n+1}, T_2^{n+1} x_{n+1}\right) + d\left(T_2^{n+1} x_{n+1}, T_2 x_{n+1}\right) \\ &\le d\left(x_{n+1}, T_2^{n+1} x_{n+1}\right) + k_2 d\left(T_2^n x_{n+1}, x_{n+1}\right). \end{aligned} \tag{3.29}$$

Taking the limit in (3.29) as $n \to \infty$ and using (3.19), (3.28), we find that

$$\lim_{n\to\infty} d\left(x_n, T_2 x_n\right) = 0.$$

Since T_3 is an asymptotically nonexpansive mapping, we have

$$\begin{aligned} d\left(x_{n+1}, T_3^n x_{n+1}\right) &\le d\left(x_{n+1}, x_n\right) + d\left(T_3^n x_{n+1}, T_3^n x_n\right) + d\left(T_3^n x_n, x_n\right) \\ &\le d\left(x_{n+1}, x_n\right) + k_n d\left(x_{n+1}, x_n\right) + d\left(T_3^n x_n, x_n\right) \end{aligned}$$

$$\leq \left(1+k_n\right) d\left(x_{n+1}, x_n\right)+d\left(T_3^n x_n, x_n\right). \tag{3.30}$$

Then, from (3.11), (3.23) and (3.30), we obtain

$$\lim_{n\to\infty} d\left(x_{n+1}, T_3^n x_{n+1}\right)=0. \tag{3.31}$$

Furthermore, we have

$$\begin{aligned} d\left(x_{n+1}, T_3 x_{n+1}\right) &\leq d\left(x_{n+1}, T_3^{n+1} x_{n+1}\right)+d\left(T_3^{n+1} x_{n+1}, T_3 x_{n+1}\right) \\ &\leq d\left(x_{n+1}, T_3^{n+1} x_{n+1}\right)+k_3 d\left(T_3^n x_{n+1}, x_{n+1}\right). \end{aligned} \tag{3.32}$$

Taking the limit in (3.32) as $n\to\infty$ and using (3.11) and (3.31), we find that

$$\lim_{n\to\infty} d\left(x_n, T_3 x_n\right)=0.$$

This implies that $\lim_{n\to\infty} d(x_n, T_1 x_n)=\lim_{n\to\infty} d(x_n, T_2 x_n)=\lim_{n\to\infty} d(x_n, T_3 x_n)=0$. This completes the proof. □

Theorem 3.3. *Let (X,d) be a complete $CAT(0)$ space, C be a nonempty, closed, and convex subset of $X, T_1, T_2, T_3 : C\to C$ be three asymptotically nonexpansive mappings with $\{k_{i_n}\}$ satisfying $\{k_{i_n}\}\subset[1,\infty)$, $\sum_{n=1}^{\infty}(k_{i_n}-1)<\infty, i=1,2,3$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences of real numbers in $[0,1]$, satisfying*

$$0<\liminf_{n\to\infty}\alpha_n\leq\limsup_{n\to\infty}\alpha_n<1, 0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1, 0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1.$$

For a given $x_1\in C$, consider the sequence $\{x_n\}$ defined by (1.2). Then $\{x_n\}$ Δ - converges to a common fixed point of T_1, T_2 and *T_3.*

Proof. It follows from Proposition 3.2 that

$$\lim_{n\to\infty} d(x_n, T_1 x_n)=\lim_{n\to\infty} d(x_n, T_2 x_n)=\lim_{n\to\infty} d(x_n, T_3 x_n)=0.$$

We let $\omega_\omega(x_n):=\cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We need to prove that $\omega_\omega(x_n)\subset\mathcal{F}=F(T_1)\cap F(T_2)\cap F(T_3)$. Let $u\in\omega_\omega(x_n)$. Then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$. By Proposition 2.5 and Proposition 2.7 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ - $\lim_{n\to\infty} v_n=v\in C$. We also have $\lim_{n\to\infty} d(v_n, T_i v_n)=0, i=1,2,3.$ Then by Proposition 2.12, we have $v\in F(T_i)$. Since $v\in F(T_i)$, then $\lim_{n\to\infty} d(x_n, v)$ exists. By Proposition 2.6, we have $u=v$. Therefore $\omega_\omega(x_n)\subset F(S_i)$. Hence $\omega_\omega(x_n)\subset\mathcal{F}=F(T_1)\cap F(T_2)\cap F(T_3)$. To show that the sequence $\{x_n\}$ Δ - converges to a common fixed point of T_1, T_2 and T_3, we will show that $\omega_\omega(x_n)$ consists of exactly one point. Suppose that $\{u_n\}$ is a subsequence of $\{x_n\}$ such that $A(\{u_n\})=\{u\}$ and $A(\{x_n\})=\{x\}$. Since $u\in\omega_\omega(x_n)\subset F(T_i)$, by Proposition 3.1, $\lim_{n\to\infty} d(x_n, u)$ exists. From Proposition 2.6, we have $x=u$. Thus, $\omega_\omega(x_n)$ consists of exactly one point. By Definition 2.4, as a result $\{x_n\}$ Δ- converges to a common fixed point of T_1, T_2 and T_3. This completes the proof. □

The following result establishes some conditions for the convergence of the iteration process (1.2) to a common fixed point of three asymptotically nonexpansive mappings in $CAT(0)$ spaces.

Theorem 3.4. *Let (X,d) be a complete $CAT(0)$ space, C be a nonempty, closed, and convex subset of $X, T_1, T_2, T_3 : C \to C$ be three asymptotically nonexpansive mappings with $\{k_{i_n}\}$ satisfying $\{k_{i_n}\} \subset [1,\infty)$, $\sum_{n=1}^{\infty}(k_{i_n}-1) < \infty, i = 1,2,3$ and $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ be sequences of real numbers in $[0,1]$, satisfying*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1, 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, 0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

For a given $x_1 \in C$, consider the sequence $\{x_n\}$ defined by (1.2). Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3 if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$ where $d(x,\mathcal{F}) = \inf\{d(x,x^*) : x^* \in \mathcal{F}\}$ and $\mathcal{F} = F(T_1) \cap F(T_2) \cap F(T_3)$.

Proof. Suppose that $\{x_n\}$ converges to $p \in \mathcal{F}$. Since $p \in \mathcal{F}$, we have $d(x_n,\mathcal{F}) \le d(x_n, p)$. Since $\{x_n\}$ converges to p, it follows that $\lim_{n\to\infty} d(x_n, p) = 0$. Therefore, $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$, which implies that $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0.$

Conversely, suppose that $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0$. We will show that $\lim_{n\to\infty} d(x_n, p) = 0$ with $p \in \mathcal{F}.$ For all $x^* \in \mathcal{F}$, by Proposition 3.1, we obtain

$$d(x_{n+1}, x^*) \le (1 + (3u_n + 3u_n^2 + u_n^3))d(x_n, x^*). \tag{3.33}$$

From this, we can deduce that $d(x_{n+1},\mathcal{F}) \le (1 + (3u_n + 3u_n^2 + u_n^3))d(x_n,\mathcal{F}).$ Also, by Proposition 2.8, we have $\lim_{n\to\infty} d(x_n,\mathcal{F})$ exists. In combination with $\liminf_{n\to\infty} d(x_n,\mathcal{F}) = 0$, it follows that $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0.$ Next, we claim that $\{x_n\}$ is a Cauchy sequence in C. Indeed, put $a_n = 3u_n + 3u_n^2 + u_n^3.$ Since $\sum_{n=1}^{\infty} u_n < \infty$, we conclude $\sum_{n=1}^{\infty} a_n < \infty.$ By using the inequality $1 + x \le e^x$ for all $x \ge 0$, from (3.33), we have $d(x_{n+1}, x^*) \le (1 + a_n)d(x_n, x^*) \le e^{a_n} d(x_n, x^*).$ This implies that

$$\begin{aligned} d(x_{m+n}, x^*) &\le e^{a_{m+n-1}} d(x_{m+n-1}, x^*) \le e^{a_{m+n-1}} e^{a_{m+n-2}} d(x_{m+n-2}, x^*) \\ &\le e^{a_{m+n-1}} e^{a_{m+n-2}} \dots e^{a_n} d(x_n, x^*) = e^{\sum_{k=n}^{m+n-1} a_k} d(x_n, x^*). \end{aligned} \tag{3.34}$$

Since $\sum_{n=1}^{\infty} a_n < \infty$, there exists $M \ge 0$ such that $e^{\sum_{k=n}^{m+n-1} a_k} = M$. Therefore, by using (3.34), we obtain $d(x_{m+n}, x^*) \le Md(x_n, x^*).$ Let $\epsilon > 0$, since $\lim_{n\to\infty} d(x_n,\mathcal{F}) = 0$, there exists there exists a positive integer n_0 such that for all $n \ge n_0$, we conclude $d(x_n, \mathcal{F}) \le \frac{\epsilon}{M+1}$. Thus, $d(x_n, \mathcal{F}) = \inf\{d(x_n, x^*); x^* \in \mathcal{F}\} \le \frac{\epsilon}{M+1}$. Hence, there exists $q \in \mathcal{F}$ such that $d(x_n, q) \le \frac{\epsilon}{M+1}$. Now, for all $m, n \ge n_0$, we get

$$d(x_{m+n}, x_n) \le d(x_{m+n}, q) + d(x_n, q) \le Md(x_n, q) + d(x_n, q)$$

$$= (M+1)d(x_n, q) \le (M+1)\frac{\epsilon}{M+1} = \epsilon.$$

Thus, $\{x_n\}$ is a Cauchy sequence in C. In this case, $\{x_n\}$ converges to $p \in C$. Combining this with $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, we find $d(p, \mathcal{F}) = 0$. As a result $p \in \mathcal{F}$. □

Theorem 3.5. *Let (X, d) be a complete $CAT(0)$ space, C be a nonempty, closed, and convex subset of $X, T_1, T_2, T_3 : C \to C$ be three asymptotically nonexpansive mappings with $\{k_{i_n}\}$ satisfying $\{k_{i_n}\} \subset [1, \infty)$, $\sum_{n=1}^{\infty}(k_{i_n} - 1) < \infty, i = 1,2,3$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be sequences of real numbers in $[0,1]$, satisfying*

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1, 0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1, 0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

For a given $x_1 \in C$, consider the sequence $\{x_n\}$ defined by (1.2) and T_1, T_2, T_3 satisfying Conditions (C). Then $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3.

Proof. According to the proof in Theorem 3.4, we find that $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. From Condition *(C)*, we obtain

$$\lim_{n\to\infty} f(d(x_n, \mathcal{F})) \le \lim_{n\to\infty} \max\{d(x_n, T_1x_n), d(x_n, T_2x_n), d(x_n, T_3x_n)\}.$$

Combining this with Proposition 3.2, we have $\lim_{n\to\infty} f(d(x_n, \mathcal{F})) = 0$. Since $f : [0;\infty) \to [0;\infty)$ a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0.$ Therefore, $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0.$ By Theorem 3.4, We conclude that $\{x_n\}$ converges strongly to a common fixed point of T_1, T_2 and T_3. This completes the proof. □

Remark 3.6. In Theorem 3.3, Theorem 3.4 and Theorem 3.5, by choosing $T_1 = T_2 = T_3 = T$, we present several results regarding Δ-convergence and strong convergence in the paper (Yambangwai & Thianwan, 2021).

Next, we provide an example to illustrate the convergence of the iteration process (1.2) to a common fixed point of three asymptotically nonexpansive mappings.

Example 3.7. Let $X = \mathbb{R}$ be a metric space with $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$ and $C = [-1,1]$. Consider three mappings $T_1, T_2, T_3 : C \to C$ defined by:

$$T_1x = \begin{cases} -2\sin\dfrac{x}{2}\ , x \in [0,1], \\ 2\sin\dfrac{x}{2}, \mathrm{x} \in [\text{-}1,0) \end{cases},\ T_2x = \begin{cases} x\ , x \in [0,1], \\ -x,\ x \in [-1,0) \end{cases} \text{ and } T_3x = \sin x.$$

From Example 2.10, we have T_1, T_2, T_3 are three asymptotically nonexpansive mappings $k_n = 1, n \in \mathbb{N}^*$. We have $\mathcal{F} = F(T_1) \cap F(T_2) \cap F(T_3) = \{0\}$.

Consider $\alpha_n = \dfrac{n}{2n+1}, \beta_n = \dfrac{n}{3n+1}, \gamma_n = \dfrac{n}{4n+1}$ for each $n \geq 1$ satisfying the Conditions of Theorem 3.3. Therefore, the iteration process defined by (1.2) converges to $0 \in \mathcal{F}$. The numerical results for the iteration process (1.2) are presented in the following table:

Table 1. The convergence data of the iteration process (1.2).

n	Dãy (1.2)
1	0.500000
2	0.166983
3	0.033482
4	0.004785
5	0.000532
6	0.000048
7	0.000004
8	0.000000

Table 1 shows that with $n = 8$, the iteration process defined by (1.2) converges to $0 \in \mathcal{F}$.

Finally, the following example shows that the iteration process (1.2) converges to a common fixed point of three asymptotically nonexpansive mappings faster than the iteration process in (Sabri, 2025).

Example 3.8. Let $X = \mathbb{R}$ be a metric space with $d(x, y) = |x - y|$, $x, y \in \mathbb{R}$ and $C = [-1,1]$. Consider three mappings $T_1, T_2, T_3 : C \to C$ defined by

$$T_1x = \frac{x}{2},\ T_2x = \begin{cases} x\ , x \in [0,1], \\ -x,\ x \in [-1,0) \end{cases} \text{ and } T_3x = \sin x.$$

From Example 2.10, we have T_1, T_2, T_3 are three asymptotically nonexpansive mappings $k_n = 1, n \in \mathbb{N}^*$. We have $\mathcal{F} = F(T_1) \cap F(T_2) \cap F(T_3) = \{0\}$.

Consider $\alpha_n = \dfrac{n}{2n+1}, \beta_n = \dfrac{n}{3n+1}, \gamma_n = \dfrac{n}{4n+1}$ for each $n \geq 1$ satisfying the Conditions of Theorem 3.3. Therefore, the iteration process defined by (1.2) converges to $0 \in \mathcal{F}$. Next, by direct computation, we compare the convergence of the proposed iteration process (1.2) with the iteration process (1) in (Sabri, 2025). First, we recall the iteration process in (Sabri, 2025) as follows:

$$\begin{cases} v_n = \gamma_n T_3^n u_n + (1-\gamma_n)u_n, \\ \mathrm{w}_n = \beta_n T_2^n v_n + (1-\beta_n)T_3^n v_n, \\ u_{n+1} = T_1^n \mathrm{w}_n, \end{cases} \tag{3.35}$$

With the three mappings T_1, T_2, T_3 as above, the iteration process (1.2) and (3.35) converges to $0 \in \mathcal{F}$. The convergence results of the two iteration processes are presented in the following table:

Table 2. The convergence data of the iteration process (1.2) and (3.35) converges to the common fixed point with $x_1 = u_1 = 0.5$

n	Dãy (1.2)	Dãy (3.35)
1	0.500000	0.500000
2	0.196032	0.413238
3	0.020394	0.285787
4	0.000509	0.177006
5	0.000003	0.102758
6	0.000000	0.057391
…	...	….
23	0.000000	0.000001
24	0.000000	0.000000

From the table, we observe that the iteration process (1.2) converges to the common fixed point after 6 steps, whereas the iterative sequence (3.35) converges after 24 steps. Therefore, the iteration process (1.2) converges to the common fixed point faster than the iteration process (3.35).

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