TRANSITION OF SPIRAL SOLUTIONS ACCORDING TO THE TIME AND SPACE STEPS DISCRETIZATION OF REACTION-DIFFUSION SYSTEM OF FITZHUGH-NAGUMO TYPE

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Abstract

Spiral solutions or spiral waves can be found in many natural systems. Spiral waves were observed in studies about the potential in brain and heart cells. Their appearance in the human heart is a presentation of arrhythmia. The paper showed how to create spiral solutions of diffusion-reaction system of FitzHugh-Nagumo type and the transition of spiral solutions according to the time step and space step discretization of finite difference method. Decreasing the value of space step discretization makes the spiral wave grow bigger, but if the value of time step discretization is increased at the same given space step, the finite difference method will be explosive, meaning that spiral wave no longer exists.

Keywords: *Reaction-diffusion equations of FitzHugh-Nagumo, space step discretization, time step discretization.*

SỰ THAY ĐỔI CỦA NGHIỆM XOẮN ỐC ĐỐI VỚI CÁC BƯỚC THỜI GIAN VÀ KHÔNG GIAN KHÁC NHAU CỦA PHƯƠNG TRÌNH PHẢN ỨNG - KHUẾCH TÁN DẠNG FITZHUGH-NAGUMO

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Tóm tắt

Nghiệm xoắn ốc hay sóng xoắn ốc có thể tìm thấy ở nhiều nơi trong thực tiễn. Các sóng xoắn ốc được quan sát khi nghiên cứu điện thế của các tế bào não và tim. Sự xuất hiện của chúng ở tim người là dấu hiệu của sự rối loạn nhịp tim. Bài báo đã đưa ra được cách tạo các nghiệm dạng xoắn ốc của hệ phương trình phản ứng - khuếch tán FitzHugh-Nagumo và sự thay đổi của nghiệm xoắn ốc phụ thuộc vào việc chọn bước thời gian và không gian của phương pháp sai phân hữu hạn. Việc giảm giá trị của bước không gian làm cho sóng xoắn ốc to dần, còn nếu làm tăng giá trị của bước thời gian ở cùng một bước không gian cho trước thì phương pháp sai phân hữu hạn bị lỗi (nổ), đồng nghĩa với nghiệm xoắn ốc không còn tồn tại nữa.

Từ khóa: *Phương trình phản ứng-khuếch tán FitzHugh-Nagumo, bước không gian, bước thời gian, nghiệm xoắn ốc.*

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1. Introduction

The FitzHugh-Nagumo model is known as a simplified two-dimensional model from the famous system of Hodgkin-Huxley (Hodgkin et al., 1952; Nagumo et al., 1962; Izhikevich, 2007; Ermentrout, 2009; Keener et al., 2009; Murray, 2010). Although the model is simple, it has many remarkable analytical results and retains the properties and biological significance. This model is made up of two equations of two variables u and v. The first variable is the fast variable, called the active variable, which represents the voltage of the cell membrane. The second one is the slow variable, which represents some time-dependent physical quantity, such as the electrical conductivity of the flow of ions across the cell membrane. The FitzHugh-Nagumo system is represented by the following system, using the notation as in (Ambrosio et al., 2012; Ambrosio et al., 2013):

$$\begin{cases} \varepsilon \dfrac{du}{dt} = f(u) - v, \\ \dfrac{dv}{dt} = au - bv + c, \end{cases} \qquad (1)$$

where a, b and c are constants (a and b are positive), $0 < \varepsilon < 1$, $t \in \mathbb{R}^+$ is the time and $f(u) = -u^3 + 3u$.

However, this system is not strong enough to reflect the propagation of cell voltage in space (along the cell body), so the cable equations are used here by adding the Laplace operator to the system (1) as follows:

$$\begin{cases} \varepsilon \dfrac{du}{dt} = \varepsilon u_t = f(u) - v + d\Delta u, \\ \dfrac{dv}{dt} = v_t = au - bv + c, \end{cases} \qquad (2)$$

where $u = u(x,t),\ v = v(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+,\ d$ is positive constant, Δu is the Laplace operator of u, $\Omega \subset \mathbb{R}^N$ is a regular bounded open set and with Neumann zero flux boundary conditions, N is a positive integer.

This system consists of two parabolic nonlinear partial differential equations, showing a wide variety of physiologically voltage-related shapes and phenomena of the cell membrane (Ambrosio et al., 2012; Ambrosio et al., 2013). Note that the first equation, also known as the cable equation, describes the flow of potentials along the body of a cell (Hodgkin et al., 1952; Izhikevich, 2007; Ermentrout, 2009). This system has been studied widely, but there is no specific study on the change of its spiral solution with different space and time steps. Spirals or spiral waves can be found in many places in practice. Spiral images are found in many applications. Spiral waves are observed when studying the electrical potential of brain and heart cells. In the heart, if the voltage wave has these patterns, the function of the heart is impaired, which is related to the problem of arrhythmia (Murray, 2010). In addition, the same results were found in the heart of rabbits, in the cerebral cortex of rats, and in the hearts of sheep. In particular, their presence in the human heart is a sign of arrhythmia. If cells in the heart's system have the same spiral wave at a certain time, it will obviously have a significant effect on the functioning of the heart. Therefore, the study of the change of the spiral solution at different time and space steps is very necessary because it helps us not only to better understand the system under consideration but also know how to control it and adjust the appearance of the spiral solution as desired.

2. Method to create a spiral solution

In this section, the results of the paper are done by numerical method, namely the finite difference method for the system (2), where

$$f(u) = -u^3 + 3u,$$
$$a = 1, b = 0.001, c = 0, \varepsilon = 0.1, d = 0.05.$$

This numerical method is implemented in C++ and the patterns are represented in Gnuplot, with

$$[0;T] \times \Omega = [0;200] \times [0;100] \times [0;100].$$

In Figure 2, there are two patterns corresponding to two solutions of the system at two different times t in the chosen space $\Omega = [0;100] \times [0;100]$. Figure 2(a) represents the solution $u(x_1, x_2, 0)$ of the system (2) at the time $t = 0$. Figure 2(b) represents $u(x_1, x_2, 190)$ at the time $t = 190$. All these solutions are called spiral solutions.

For the system (2), in order to create a spiral solution, the domain Ω is divided into four parts with almost the same area. On each of those sub-domains, we choose the initial condition as constant functions $(u(x,0),v(x,0))$ in such a way that these constant functions are regularly out of phase with each other at intervals on the normal circle of the system (1). These initial conditions can be chosen as shown in Figure 1 below, and by the finite difference method a spiral solution is generated as shown in Figure 2 (see also in Phan 2019).

$(u(x,0),v(x,0))=(0,-1)$	$(u(x,0),v(x,0))=(-1,0)$
$(u(x,0),v(x,0))=(1,0)$	$(u(x,0),v(x,0))=(0,1)$

Figure 1. Initial conditions allow the system (2) to have a spiral solution

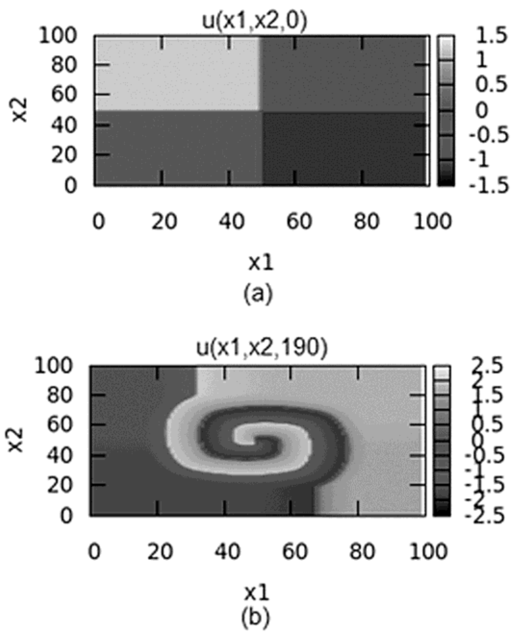

Figure 2. The solution takes the form of a spiral of system (2) corresponding to the initial conditions given in Figure 1. Figure (a) represents the solution $u(x_1,x_2,0)$ at the time $t=0$. Figure (b) represents $u(x_1,x_2,190)$ at the time $t=190$

Similarly, if the domain Ω is divided into 16 (corresponding to 64) equal parts, then the solution of the system (2) will have the form of 4 (corresponding to 16) spirals illustrated by Figure 3 (corresponding to Figure 4).

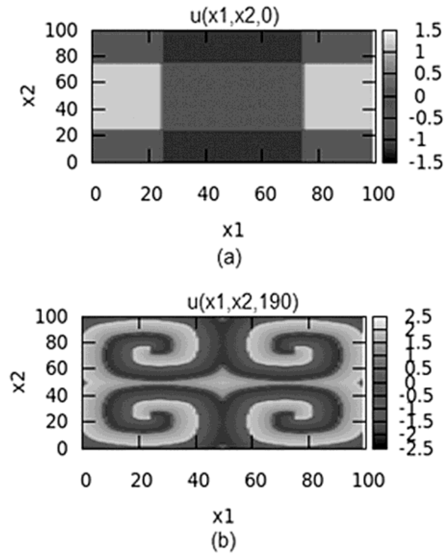

Figure 3. The solution takes the form of four spirals of (2). Figure (a) represents the solution $u(x_1,x_2,0)$ at the time $t=0$. Figure (b) represents $u(x_1,x_2,190)$ at the time $t=190$

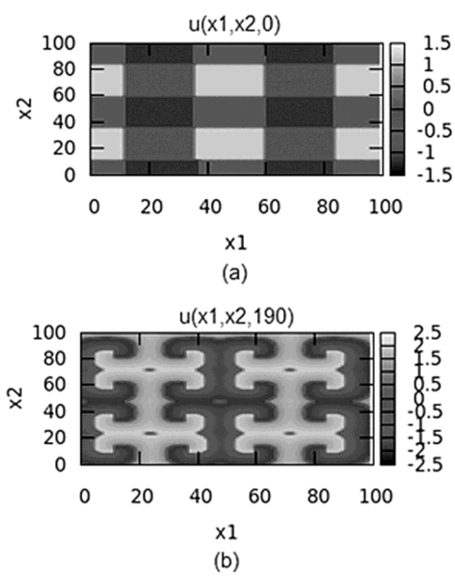

Figure 4. The solution has the form of 16 spirals of (2). Figure (a) represents the solution $u(x_1,x_2,0)$ at the time $t=0$. Figure (b) represents $u(x_1,x_2,190)$ at the time $t=190$

3. Transition of spiral solutions according to the time and space step discretization

The results in this section are further performed by the finite difference method for the system (2), in which the time and space steps are continuously changed to observe the transition of the spiral solution as well as when it disappears.

Let h and dt be the space and time steps of the finite difference method, respectively.

In Figure 5, we fix $dt = 0.01$ and change the value of the space step as follows: (a) $h = 1$, (b) $h = 0.9$, (c) $h = 0.7$. The results show that the spiral gets bigger as the space step gets smaller, which is understandable because the smaller the space step is, the more specific the solution of the problem will be at the locations where the space greater is not possible. However, if the space step takes on a smaller value, for example $h = 0.5$, the difference method fails (explosive), the spiral does not exist anymore.

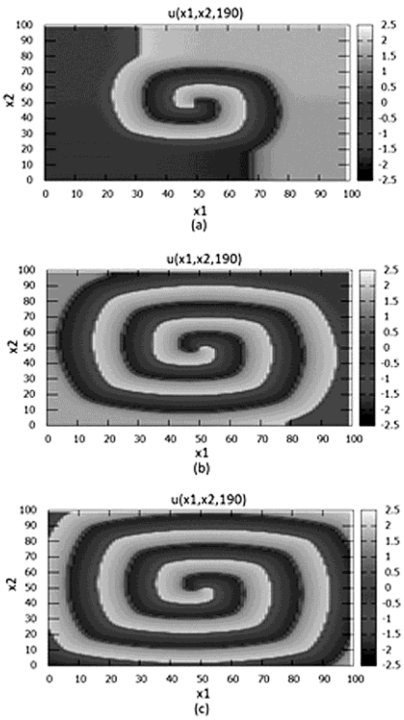

Figure 5. The solution of (2) is in the form of a spiral corresponding to the initial condition given in Figure 1, describing the solution $u(x_1, x_2, 190)$ at the time $t = 190$, the time step $dt = 0.01$ and the space step changes as follows: (a) $h = 1$, (b) $h = 0.9$, (c) $h = 0.7$

In Figure 6, we fix $h = 1$ and change the value of the time step as follows: (a) $dt = 0.0025$, (b) $dt = 0.005$, (c) $dt = 0.025$. The results show that the spiral solution gets smaller and less smooth as the time step increases. Also, if the time step takes on a larger value, for example, $dt = 0.05$, the difference method is faulty (explosive), the spiral does not exist anymore.

In Figure 7, we fix $h = 0.5$ and change the value of the time step as follows: (a) $dt = 0.0025$, (b) $dt = 0.005$. The results also show that the spiral solution does not change much, perhaps the solution becomes less smooth because the time step increases. In addition, because the space step is subdivided, the solution of the problem is more specific and the spiral wave is also larger than the case of $h = 1$. But if the time step takes on a larger value, for example $dt = 0.025$, the difference method is faulty (explosive), the spiral does not exist anymore.

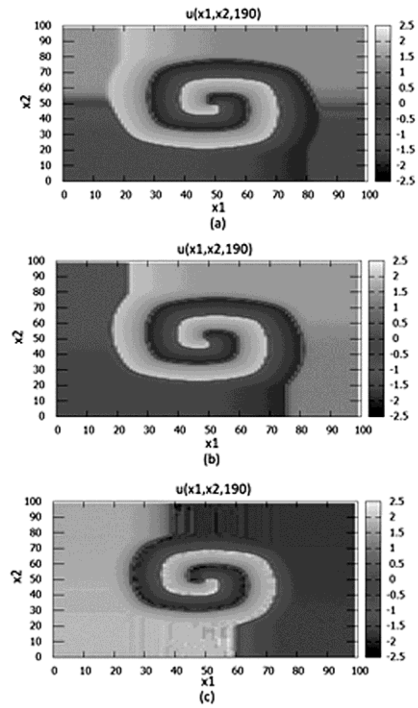

Figure 6. The solution of (2) is in the form of a spiral corresponding to the initial condition given in Figure 1, describing the solution $u(x_1, x_2, 190)$ at the time $t = 190$, the space step $h = 1$ and the time step changes as follows: (a) $dt = 0.0025$, (b) $dt = 0.005$, (c) $dt = 0.025$

In Figure 8, we fixed $h = 0.2$ and changed the value of the time step as follows: (a) $dt = 0.0001$, (b) $dt = 0.0005$. The results show that the spiral wave does not change much, perhaps the solution becomes less smooth because the time step increases. In addition, because the space step is subdivided, the solution of the problem is expressed more specifically and the spiral wave is also larger than the case of $h = 0.5$ and $h = 1$. But if the time step takes on a larger value for example

$dt = 0.0025$ then the difference method is faulty (explosive), the spiral doesn't exist anymore.

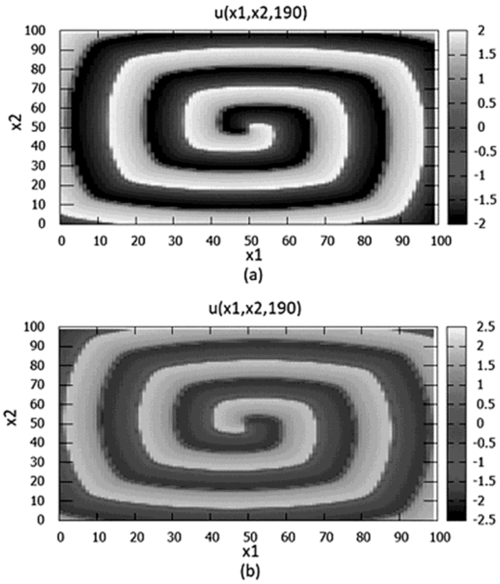

Figure 7. The solution of (2) is in the form of a spiral corresponding to the initial condition given in Figure 1, describing the solution $u(x_1, x_2, 190)$ at the time $t = 190$, the space step $h = 0.5$ and the time step changes as follows: (a) $dt = 0.0025$, (b) $dt = 0.005$

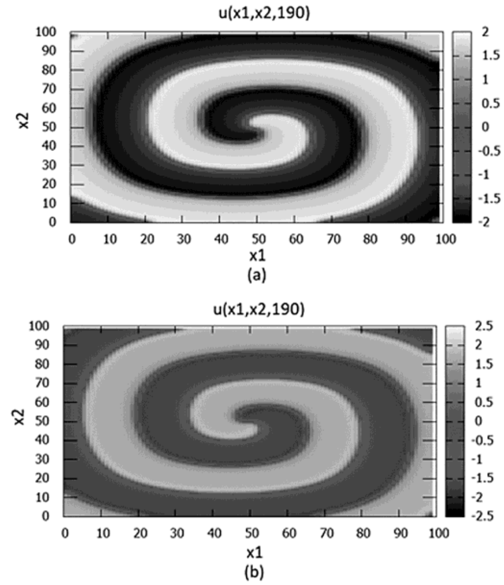

Figure 8. The solution of (2) is in the form of a spiral corresponding to the initial condition given in Figure 1, describing the solution $u(x_1, x_2, 190)$ at the time $t = 190$, the space time $h = 0.2$ and the time step changes as follows: (a) $dt = 0.0001$, (b) $dt = 0.0005$

Table 1. Transition of spiral solutions according to different time and space steps of reaction-diffusion system of FitzHugh-Nagumo type

Space step h	**Time step** dt	**Transition of the spiral solution of system (2)**
1	0.01	The spiral appears
0.9	0.01	The spiral grows bigger
0.7	0.01	The spiral grows bigger
0.5	0.01	The spiral vanishes (explosive)
1	0.0025	The spiral appears
1	0.005	The spiral becomes smaller
1	0.025	The spiral becomes smaller and less smooth
1	0.05	The spiral vanishes (explosive)
0.5	0.0025	The spiral appears, large and smooth
0.5	0.005	The spiral is less smooth
0.5	0.025	The spiral vanishes (explosive)
0.2	0.0001	The spiral appears, large and smooth
0.2	0.0005	The spiral is less smooth
0.2	0.0025	The spiral vanishes (explosive)

From the above results, it is shown that the change of the spiral solution depends on the choice of time and space steps of the finite difference method for the system (2). Decreasing the value of the space step causes the spiral wave to become larger, and if the time step is increased at the same given space step, the finite difference method is faulty (explosive), which means the spiral does not exist any more (see Table 1).

4. Conclusion

The article has shown how to create the spiral solutions of reaction -diffusion system of FitzHugh-Nagumo type and the change of the spiral solution depends on the choice of time and space steps of the finite difference method. Decreasing the value of the space step causes the spiral wave to become larger, and if the time step is increased at the same given space step, the finite difference method is faulty (explosive), which means the spiral does not exist any more. In the next paper, the author will study the synchronization of spiral solutions in the case of a complete network with non-linear coupling.

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