DETERMINE MATRICES RECEIVING GIVEN PAIRWISE DISTINCT REAL NUMBERS AS EIGENVALUES

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Abstract

The paper shows a method to give matrices that receive given pairwise distinct real numbers as eigenvalues. Thereby, multiple examples and exercises are provided with given answers. The method applied does not involve handheld calculator use.

Keywords: *Eigenvalues, similar matrices, trace of a matrix.*

XÁC ĐỊNH MA TRẬN NHẬN CÁC SỐ THỰC PHÂN BIỆT CHO TRƯỚC LÀM GIÁ TRỊ RIÊNG

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Tóm tắt

Bài viết này trình bày một phương pháp thiết lập ma trận nhận các giá trị cho trước (đôi một khác nhau) làm giá trị riêng. Từ đó, chúng ta có thể đưa ra ví dụ và bài tập xác định giá trị riêng của một ma trận với kết quả đã được chọn trước. Phương pháp trong bài viết này của chúng tôi không sử dụng máy tính cầm tay.

Từ khóa: *Giá trị riêng, ma trận đồng dạng, vết của ma trận.*

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1. Introduction

This article continues the topic of giving exercises with preselected answers in linear algebra reported by Truong et al. (2020) and Nguyen et al. (2022).

A square matrix A is said to be diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ has diagonal form. To diagonalize a matrix, we must first solve the characteristic equation to find the eigenvalues, from which we know if the matrix is diagonalizable or not. However, because the characteristic equation is a polynomial equation of degree n, so finding the solution is not easy with complex numbers or roots, causing eigenvectors become more complicated. Therefore, in the process of giving examples and test questions, it needs controlling the eigenvalues of the matrix to actively control the results of the problems. This article presents a method to give matrices that receive given pairwise distinct real numbers as eigenvalues.

The basic knowledge and symbols used in this article follow those presented in Advanced Math A3 textbook by Dinh and Vo (2019).

2. Contents

For simplicity, this article only considers square matrices with real coefficients because the expansion on any field is completely similar.

First, we prove the following results, from which we can deduce two results of Truong et al. (2020):

2.1. Theorem 1

Let $\left(a_{\text{ij}}\right)_n \in M_n\left(\mathbb{R}\right)$ be a matrix and λ_i $(i=1,...,n)$ be pairwise distinct real numbers. Then, the matrix

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{11}-\lambda_1 & a_{12}+\lambda_1 & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{11}-\lambda_2 & a_{12} & a_{13}+\lambda_2 & \cdots & a_{1,n-1} & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & a_{1n} \\ a_{11}-\lambda_{n-2} & a_{12} & a_{13} & \cdots & a_{1,n-1}+\lambda_{n-2} & a_{1n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{1n}+\lambda_{n-1} \end{pmatrix}$$

such that the sum of the elements on the same row is equal λ_n will be received $\lambda_1, \lambda_2, \ldots, \lambda_n$ as eigenvalues. In particular, the matrix

$$A=\begin{pmatrix} a & b & c & d \\ a-\lambda_1 & b+\lambda_1 & c & d \\ a-\lambda_2 & b & c+\lambda_2 & d \\ p & q & r & d+\lambda_3 \end{pmatrix}$$

such that the sum of the elements on the same row is equal λ_4 will be received $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ as eigenvalues.

Proof. We have the characteristic equation of A is

$$P(\lambda)=\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{11}-\lambda_1 & a_{12}+\lambda_1-\lambda & a_{13} & \cdots & a_{1,n-1} & a_{1n} \\ a_{11}-\lambda_2 & a_{12} & a_{13}+\lambda_2-\lambda & \cdots & a_{1,n-1} & a_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & a_{1n} \\ a_{11}-\lambda_{n-2} & a_{12} & a_{13} & \cdots & a_{1,n-1}+\lambda_{n-2}-\lambda & a_{1n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{n,n-1} & a_{1n}+\lambda_{n-1}-\lambda \end{vmatrix}=0.$$

- When $\lambda=\lambda_i$ for $i=\overline{1,n-2}$: We have $d_1=d_{i+1}$ implies $P(\lambda_i)=0$, that means $\lambda_1,\ldots,\lambda_{n-2}$ are eigenvalues of the matrix.
- When $\lambda=\lambda_n$: We have $c_1+c_2+\cdots+c_n=0$ implies $P(\lambda_n)=0$, this means λ_n is an eigenvalue of the matrix.
- Let λ' be the remaining complex solution of the characteristic equation $P(\lambda)=0$. Since

$P(\lambda)=|A-\lambda I|$

$=\left(-\lambda\right)^n+\ tr(A).\left(-\lambda\right)^{n-1}+\ \cdots\ +\det(A)$ by

$=\left(-1\right)^n.\lambda^n+\ \left(-1\right)^{n-1}.tr(A).\lambda^{n-1}+\ \cdots\ +\det(A),$

Vieta's formulas and hypothesis

$$a_{11}+a_{12}+\cdots+a_{1n}=\lambda_n,$$

we have

$$\begin{aligned}&\lambda_1+\lambda_2+\cdots+\lambda_{n-2}+\lambda_n+\lambda' \\ &=-\frac{\left(-1\right)^{n-1}.tr(A)}{\left(-1\right)^n}=tr(A) \\ &=a_{11}+\left(a_{12}+\lambda_1\right)+\cdots+\left(a_{1n}+\lambda_{n-1}\right) \\ &=\lambda_1+\lambda_2+\cdots+\lambda_{n-1}+\lambda_n.\end{aligned}$$

Implies

$$\begin{aligned}&\lambda_1+\lambda_2+\cdots+\lambda_{n-2}+\lambda_n+\lambda' \\ &=\lambda_1+\lambda_2+\cdots+\lambda_{n-2}+\lambda_n+\lambda_{n-1}.\end{aligned}$$

- Then $\lambda'=\lambda_{n-1}$, this means λ_{n-1} is also an eigenvalue of the matrix.

Hence, the matrix A receives $\lambda_1, \lambda_2, \ldots, \lambda_n$ as eigenvalues. □

Note: In the proof of the theorem above, we used the following two basic results:

i) A matrix such that the sum of all elements in the same row (or column) is equal λ will be received λ as an eigenvalue.

ii) If a matrix A receives real numbers λ_i $(i=1,...,n)$ as eigenvalues, then $Tr(A)=\sum_{i=1}^{n}\lambda_i$.

With the same proof technique as above, it is easy to obtain the following two results by Truong Thi Thuy Van et al. (2020).

2.2. Corollary 2 (Truong et al., 2020, Theorem 1, p. 90)

A matrix $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ satisfies the following conditions receiving real numbers λ_1, λ_2, λ_3 as eigenvalues:

i) The sum of all elements on the same row is equal to λ_3.

ii) $d = a-\lambda_1$, $e=b+\lambda_1$, $f=c$, $k=c+\lambda_2$.

In other words, the matrix

$$A=\begin{pmatrix} a & b & c \\ a-\lambda_1 & b+\lambda_1 & c \\ g & h & c+\lambda_2 \end{pmatrix}$$

such that the sum of elements in the same row is equal to λ_3 and receives the real numbers λ_1, λ_2, λ_3 as eigenvalues.

2.3. Corollary 3 (Truong et al., 2020, Theorem 2, p. 90)

A matrix $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ that satisfies the following conditions receiving real numbers λ_1, λ_2, λ_3 as eigenvalues::

i) The sum of the elements on the same row is equal to λ_3.

ii) $e=b+\lambda_2$, $g=a-\lambda_1$, $h=b$, $k=c+\lambda_1$.

In other words, the matrix

$$A=\begin{pmatrix} a & b & c \\ d & b+\lambda_2 & f \\ a-\lambda_1 & b & c+\lambda_1 \end{pmatrix}$$

such that the sum of elements in the same row is equal to λ_3 receives the real numbers $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.

Note: Corollary 2 and Corollary 3 have the advantage of being able to quickly put numbers in to get the exercises with the desired answer. However, Corollary 2 (respectively, Corollary 3) has the disadvantage that $a_{13}=a_{23}=c$ (respectively, $a_{12}=a_{32}=b$), leading to the matrix not yet completely natural. In addition, learners can brainstorm solutions by choosing values so that two rows are equal, as well as they can see that the sum of elements in the same row is equal. We will fix the above problem as follows:

2.4. Theorem 4

If the matrix $A=\left(a_{ij}\right)_n \in M_n\left(\mathbb{R}\right)$ receives pairwise distinct real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ as eigenvalues, then the matrix B that is obtained from the matrix A by taking the row *i* plus *k* times the row *j*, and then taking the column *j* minus *k* times the column *i*, also receives $\lambda_1, \lambda_2, \ldots, \lambda_n$ as eigenvalues.

Proof. When performing the elementary transformation taking the row i plus the row j of the matrix A, that is, multiplying to the left of the matrix A a primary matrix (obtained from the unit matrix by taking the row i plus the row j) and multiply to the right of the matrix A an elementary matrix F (obtained from the unit matrix by taking the column *j* minus *k* times the column *i*). Two matrices E and F are inverses of each other, that is, two matrices E and are similar, so that they have the same eigenvalues.

We illustrate the proof in detail in the following corollary:

2.5. Corollary 5

If the matrix $A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ receives pairwise distinct real numbers $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues, then the matrix B is obtained from the matrix A by taking the row 1 plus the row 3, and then taking column 3 minus the column *i*, also receives $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.

Proof.

We have

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

$$\xrightarrow{h_1 \to h_1+h_3}\begin{pmatrix} a+g & b+h & c+k \\ d & e & f \\ g & h & k \end{pmatrix}$$

$$\xrightarrow{c_3 \to c_3-c_1}\begin{pmatrix} a+g & b+h & c+k-a-g \\ d & e & f-d \\ g & h & k-g \end{pmatrix}=B.$$

On the other hand, it is easy to check $B=P^{-1}AP$ with $P=\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $P^{-1}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, that means A and B are two similar matrices. Hence, the matrix B also receives $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.

2.6. Algorithm for setting square matrix of level 3

We can apply Corollary 2 to establish the matrix.

$$A=\begin{pmatrix} a & b & c \\ a-\lambda_1 & b+\lambda_1 & c \\ g & h & c+\lambda_2 \end{pmatrix}$$

that receives $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.

Step 1: Write arbitrary values a,b,c in the following positions.

$$\begin{pmatrix} a & b & c \\ * & * & c \\ * & * & * \end{pmatrix},$$

such that $a+b+c=\lambda_3$.

Step 2: Calculate $a-\lambda_1, b+\lambda_1, c+\lambda_2$ to write in the next position.

$$\begin{pmatrix} a & b & c \\ a-\lambda_1 & b+\lambda_1 & c \\ * & * & c+\lambda_2 \end{pmatrix}.$$

Step 3: Fill in the remaining two values arbitrarily such that the sum of all the elements in the row 3 is equal to λ_3.

$$A=\begin{pmatrix} a & b & c \\ a-\lambda_1 & b+\lambda_1 & c \\ g & h & c+\lambda_2 \end{pmatrix}.$$

This matrix receives $\lambda_1, \lambda_2, \lambda_3$ as eigenvalue. However, we can perform the following additional transformations to get a matrix whose elements are less dependent on each other.

Step 4: Take the row 1 plus the row 3, and then take the column 3 minus the column 1, the resulting matrix also receives $\lambda_1, \lambda_2, \lambda_3$ as eigenvalues.

Note: We can also perform consecutive pairs of transformations on rows and columns of the matrix $A=\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$ by taking the row *i* plus *k* times the row *j* and then taking the column *j* minus *k* times the column *i*, but this process is quite time consuming.

2.7. Illustration example

One example is provided for teachers to create a test when teaching in class. For example, teachers would like students to find the eigenvalues of a square matrix of level 3 with the given answers are 3, 4, 9, the essential steps should be as follows:

Step 1: Write arbitrary values a,b,c in the following positions.

$$\begin{pmatrix} a & b & c \\ * & * & c \\ * & * & * \end{pmatrix},$$

such that $a+b+c=\lambda_3=9$.

For example

$$\begin{pmatrix} -2 & 5 & 6 \\ * & * & 6 \\ * & * & * \end{pmatrix}.$$

Step 2: Calculate

$$a-\lambda_1 = -2-3 = -5,$$
$$b+\lambda_1 = 5+3 = 8,$$
$$c+\lambda_2 = 6+4 = 10$$

to write in the next position.

$$\begin{pmatrix} -2 & 5 & 6 \\ -5 & 8 & 6 \\ * & * & 10 \end{pmatrix}.$$

Step 3: Fill in the remaining two values arbitrarily such that the sum of all the elements in the row 3 is equal to $\lambda_3 = 9$.

For example

$$A = \begin{pmatrix} -2 & 5 & 6 \\ -5 & 8 & 6 \\ 8 & -9 & 10 \end{pmatrix}.$$

Step 4: Take the row 1 plus the row 3, and then take the column 3 minus the column 1.

$$A = \begin{pmatrix} -2 & 5 & 6 \\ -5 & 8 & 6 \\ 8 & -9 & 10 \end{pmatrix}$$
$$\xrightarrow{h_1 \to h_1 + h_3} \begin{pmatrix} 6 & -4 & 16 \\ -5 & 8 & 6 \\ 8 & -9 & 10 \end{pmatrix}$$
$$\xrightarrow{c_3 \to c_3 - c_1} \begin{pmatrix} -6 & -4 & 10 \\ -5 & 8 & 11 \\ 8 & -9 & 2 \end{pmatrix} = B.$$

Two matrices A and B both receive 3, 4, 9 as eigenvalues.

3. Conclusion

This article has presented one more method to help teachers be more active when making problems to find the eigenvalues of the matrix with pre-selected values. In addition, mastering the constraints can extend the theorems by:

- Change the condition i) to be stronger to "The columns (or rows) of the matrix $A-\lambda_i I$ are linearly dependent".
- Extend the results to the case of matrices of level *n*.
- Consider the case of a characteristic polynomial with multiple roots.
- Change the matrix P in Step 4.

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