DRINFEL'D ASSOCIATOR AND RELATIONS OF SOME SPECIAL FUNCTIONS

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Abstract

We observe the differential equation $dG(z)/dz = (x_0/z + x_1/(1-z))G(z)$ *in the space of power series of noncommutative indeterminates* x_0, x_1*, where the coefficients of* $G(z)$ *are holomorphic functions on the simply connected domain* $\mathbb{C} \setminus [(-\infty, 0) \cup (1, +\infty)]$. *Researches on this equation in some conditions turn out different solutions which admit Drinfel'd associator as a bridge. In this paper, we review representations of these solutions by generating series of some special functions such as multiple harmonic sums, multiple polylogarithms and polyzetas. Thereby, relations in explicit forms or asymptotic expansions of these special functions from the bridge equations are deduced by identifying local coordinates.*

Keywords: *Drinfel'd associator, multiple harmonic sums, multiple polylogarithms, polyzetas, special functions.*

LIÊN HỢP DRINFEL'D VÀ QUAN HỆ CỦA MỘT SỐ HÀM ĐẶC BIỆT

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Tóm tắt

Chúng tôi quan sát phương trình vi phân $dG(z)/dz = (x_0/z + x_1/(1-z))G(z)$ *trong không gian các chuỗi lũy thừa của các phần tử không giao hoán* x_0, x_1, *trong đó các hệ số của* $G(z)$ *là các hàm chỉnh hình trên miền đơn liên* $\mathbb{C} \setminus [(-\infty, 0) \cup (1, +\infty)]$. *Những nghiên cứu xung quanh phương trình này trong một số điều kiện khác nhau cho ta những nghiệm khác nhau và liên hợp Drinfiel'd là một cầu nối giữa chúng. Trong bài báo này, chúng tôi tổng quan lại việc biểu diễn các trường hợp nghiệm thông qua các hàm sinh của các hàm đặt biệt như tổng điều hòa bội, hàm polylogarit bội và chuỗi zeta bội. Từ các phương trình cầu nối, chúng tôi rút ra được các quan hệ dưới dạng tường minh hoặc khai triển tiệm cận của các các hàm đặc biệt này bằng cách đồng nhất các tọa độ địa phương.*

Từ khóa: *Liên hợp Drinfel'd, tổng điều hòa bội, hàm polylogarit bội, chuỗi zeta bội, tổng điều hòa bội.*

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1. Introduction

Let $\mathbb{C}_*^n := \{(z_1,\ldots,z_n)\in\mathbb{C}^n |\ z_i \neq z_j \text{ for } i\neq j\}$ and $\mathcal{H}(\mathbb{C}_*^n)$ denotes the ring of holomorphic functions over the universal covering of $\mathbb{C}_*^n$, denoted by $\mathbb{C}_*^n$. Using $\mathcal{T}_n := \{t_{ij}\}_{1\le i<j\le n}$ as an alphabet, Knizhnik & Zamolodchikov (1984) defined a noncommutative first order differential equation acting in the ring $\mathcal{H}(\mathbb{C}_*^n)\langle\langle\mathcal{T}_n\rangle\rangle$,

$$dG(z) = \Omega_n(z)G(z), \tag{1.1}$$

where $\Omega_n := \sum_{1\le i<j\le n}\frac{t_{ij}}{2i\pi} d\log(z_i - z_j)$.

For example, with $n=2$, one has $\mathcal{T}_2 = \{t_{12}\}$ and a solution of the equation $dG(z) = \Omega_2 G(z)$, where $\Omega_2 = \frac{t_{12}}{2i\pi} d\log(z_1 - z_2)$, is

$$G(z_1,z_2) = \exp(\frac{t_{12}}{2i\pi}\log(z_1-z_2)) = (z_1-z_2)^{t_{12}/2i\pi} \in \mathcal{H}(\mathbb{C}_*^2)\langle\langle\mathcal{T}_2\rangle\rangle.$$

In the case $n=3$, the equation

$$dG(z) = \frac{1}{2i\pi}\Big(t_{12}\frac{dz}{z} - t_{23}\frac{dz}{1-z}\Big)G(z) \tag{1.2}$$

is appplied in the ring $\mathcal{H}(\mathbb{D})\langle\langle t_{12}, t_{23}\rangle\rangle$, where

$$\mathbb{D} := \mathbb{C} \setminus [(-\infty,0)\cup(1,+\infty)]. \tag{1.3}$$

By taking $x_0 := \frac{t_{13}}{2i\pi}, x_1 := \frac{-t_{23}}{2i\pi}$, equation (1.2) can be rewritten as follows

$$\frac{dG(z)}{dz} = \Big(\frac{x_0}{z} + \frac{x_1}{1-z}\Big)G(z), \tag{1.4}$$

and more shortly $dG(z) = (\omega_0(z)x_0 + \omega_1(z)x_1)G(z)$ by using the two differential forms

$$\omega_0(z) := \frac{dz}{z} \text{ and } \omega_1(z) := \frac{dz}{1-z}. \tag{1.5}$$

The resolution of (1.4) uses the so-called

Chen series, of ω_0 and ω_1 along a path $z_0 \rightsquigarrow z$ on $\mathcal{D}$, defined by (Cartier, 1987):

$$C_{z_0\rightsquigarrow z} := \sum_{w\in X^*}\alpha_{z_0}^z(w)w \in \mathcal{H}(\mathbb{D})\langle\langle X\rangle\rangle, \tag{1.6}$$

where X^* denotes the free monoid generated by the alphabet X (equipping the empty word as the neutral element) and, for a subdivision $(z_0, z_1\ldots,z_k,z)$ of $z_0 \rightsquigarrow z$ and the coefficient $\alpha_{z_0}^z(w)\in\mathcal{H}(\mathbb{D})$ is defined, for any $w = x_{i_1}\cdots x_{i_k}\in X^*$, as follows

$$\alpha_{z_0}^z(w) = \int_{z_0}^z \omega_{i_1}(z_1)\ldots\int_{z_0}^{z_{k-1}}\omega_{i_k}(z_k),$$
$$\alpha_{z_0}^z(1_{X^*}) = 1_{\mathcal{D}}.$$

The series $C_{z_0\rightsquigarrow z}$ is group-like (Ree, 1958), which implies that there exists a primitive series $L_{z_0\rightsquigarrow z}$ such that

$$e^{L_{z_0\rightsquigarrow z}} = C_{z_0\rightsquigarrow z}. \tag{1.7}$$

In (Drinfel'd, 1990), Drinfel'd is essentially interested in solutions of (1.4) over the interval $(0;1)$ and, using the involution $z\mapsto 1-z$, he stated (1.4) admits a unique solution G_0 (resp. G_1) satisfying asymptotic forms

$$G_0(z)_{z\to 0} z^{x_0} \text{ and } G_1(z)_{z\to 1}(1-z)^{-x_1}. \tag{1.8}$$

Moreover, G_0 and G_1 are group-like series then there is a unique group-like series $\Phi_{KZ}\in\mathbb{R}\langle\langle X\rangle\rangle$, Drinfel'd series (so-called Drinfel'd associator), such that

$$G_0 = G_1\Phi_{KZ}. \tag{1.9}$$

After that, via a regularization based on representation of the chord diagram algebras Thang & Murakami (1996) expressed the divergent coefficients of Φ_{KZ} as linear combinations of Multiple-Zeta-Value (or polyzetas) defined for each composition $(s_1,\ldots,s_r)\in\mathbb{N}_{\ge1}^r, s_1\ge 2,$ as follows

$$\zeta(s_1,\ldots,s_r) := \sum_{n_1>\ldots>n_r\ge1}\frac{1}{n_1^{s_1}\ldots n_r^{s_r}}. \tag{1.10}$$

In other words, these polyzetas can be reduced by the limit at $z=1$ of multiple polylogarithms or at $N\to\infty$ of multiple harmonic sums, respectively defined on each multi-index $(s_1,\ldots,s_r)\in\mathbb{N}_{\ge1}^r, r\ge1$, and $z\in\mathbb{C}, |z|<1, n\in\mathbb{N}$, as follows

$$\mathrm{Li}_{s_1,\dots,s_r}(z) := \sum_{n_1>\dots>n_r\geq 1} \frac{z^{n_1}}{n_1^{s_1}\dots n_r^{s_r}}, \qquad (1.11)$$

$$\mathrm{H}_{s_1,\dots,s_r}(n) := \sum_{n_1>\dots>n_r\geq 1}^{n} \frac{1}{n_1^{s_1}\dots n_r^{s_r}}. \qquad (1.12)$$

Moreover, the multiple harmonic sums can be viewed as coefficients of generating series of the multiple polylogarithm for each multi-index

$$(1-z)^{-1}\mathrm{Li}_{s_1,\dots,s_r}(z) = \sum_{n\geq 1} \mathrm{H}_{s_1,\dots,s_r}(n) z^n. \qquad (1.13)$$

In this work, we review a method to construct relations of the special functions by following equation (1.9). The generating series of the special functions are group-like series to review simultaneously the essential steps to furnish G_0 and Φ_{KZ} which follows related equations in asymptotic expansion forms and then an equation bridging the algebraic structures of converging polyzetas.

2. Algebras of shuffle and quasi-shuffle products

The above special functions are compatible with shuffle and quasi-shuffle structures. In order to represent these properties more clearly, we correspond each multi-index $(s_1,\dots,s_r)\in\mathbb{N}_{\geq 1}^r, r\geq 1$ to words generated by the two alphabets $X=\{x_0,x_1\}$ and $Y=\{y_k\}_{k\geq 1}$ as follows

$$(s_1,\dots,s_r) \leftrightarrow x_0^{s_1-1}x_1\dots x_0^{s_r-1}x_1 \in X^*x_1 \underset{\pi_Y}{\overset{\pi_X}{\rightleftharpoons}} y_{s_1}\dots y_{s_r} \in Y^*, \qquad (2.1)$$

Where X^* and Y^* respectively denote the free monoids of words generated by the alphabets X and Y with the empty words 1_{X^*} and 1_{Y^*} (sometime use 1 in common) as the neutral elements. This section reviews two structures of shuffle and quasi-shuffle algebras compatible with the special functions introduced above.

2.1. Bi-algebras in duality

By taking formal sums of words, we can extend the monoids X^* and Y^* to the $\mathbb{Q}$-modules, denoted by $\mathbb{Q}\langle X\rangle$ and $\mathbb{Q}\langle Y\rangle$, which become bi-algebras with respect to the following product and co-product:

1. The associative unital concatenation, denoted by $conc$, and its co-law which is denoted by Δ_{conc} and defined for any w as follows

$$\Delta_{conc}(w) = \sum_{uv=w} u\otimes v; \qquad (2.2)$$

2. The associative commutative and unital shuffle product defined, for any $x,y\in X$ and $u,v\in X^*$, by the recursion

$$u ш 1_{X^*} = 1_{X^*} ш u = u,$$
$$xu ш yv = x(u ш yv) + y(xu ш v), \qquad (2.3)$$

or equivalently, by its coproduct (which is a morphism for concatenations) defined, for each letter $x\in X$, as follows

$$\Delta_ш x = 1_{X^*}\otimes x + x\otimes 1_{X^*}. \qquad (2.4)$$

According to the Radford theorem (Radford, 1979), Lyn$X$ forms a pure transcendence basis of the $\mathbb{Q}$-shuffle algebras, graded in length of word, $(\mathbb{Q}\langle X\rangle, ш, 1_{X^*})$ (Reutenauer, 1993). Similarly, the $\mathbb{Q}$-module $\mathbb{Q}\langle Y\rangle$ is also equipped with the associative commutative and unital stuffle product defined, for $u,v,w\in Y^*$ and $y_i,y_j\in Y$, by

$$w ж 1_{Y^*} = 1_{Y^*} ж w = w,$$
$$y_i u ж y_j v = y_i(u ж y_j v) + y_j(y_i u ж v) + y_{i+j}(u ж v).$$

It can be dualized according to $y_k\in Y$

$$\Delta(y_k) := y_k\otimes 1_{Y^*} + 1_{Y^*}\otimes y_k + \sum_{i+j=k} y_i\otimes y_j$$

which is also a $conc$ −morphism and the $\mathbb{Q}$-stuffle algebra $(\mathbb{Q}\langle Y\rangle, Ж, 1_{Y^*})$

admits the set of Lyndon words, denoted by Lyn$Y$, as a pure transcendence basis (Hoang, 2013b; Chien et al., 2015). This algebra is graded in weight defined by taking sum of all index of letters in a word. For example, the weight of the word $w=y_{s_1}\dots y_{s_r}$ is $s_1+\dots+s_r$.

Note that, the stuffle product defined here just acts on the monoid generated by alphabet Y but the shuffle product can be applied for any alphabet.

We will use $\mathcal{X}$ as a general alphabet used for shuffle product and A as a field extension of $\mathbb{Q}$.

Definition 2.1. Let $A\langle\langle\mathcal{X}\rangle\rangle$, $S \in A\langle\langle Y\rangle\rangle$ be the sets of formal series extended from $A\langle X\rangle$ and $A\langle Y\rangle$ respectively. Then

i. S is said to be a *group-like series* if and only if $\langle S 1_{\mathcal{X}^*} = 1$ and $\Delta_{ш} S = S \otimes S$ (resp. $\Delta_{ж} S = S \otimes S$).

ii. S is said to be a *primitive series* if and only if $\Delta_{ш} S = 1_{\mathcal{X}^*} \otimes S + S \otimes 1_{\mathcal{X}^*}$ (resp. $\Delta_{ж} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$).

The Lie bracket in an algebra is defined for some algebra with the product $(\cdot)$ as usual

$$[x;y] = x \cdot y - y \cdot x.$$

The following results are standard facts from works by Ree (Ree, 1958) (see also (Chien et al., 2015; Reutenauer, 1993).

Proposition 2.1.

i. The Lie bracket of two primitive elements is primitive.

ii. Let $S \in A\langle Y\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$). Then S is primitive, for $\Delta_{ж}$ (resp. Δ_{conc} and $\Delta_{ш}$), if and only if, for any $u,v \in Y^* Y$ (resp. $\mathcal{X}^*\mathcal{X}$), we get $\langle S \mid u ж v\rangle = 0$ (resp. $\langle S \mid uv\rangle = 0$ and $\langle S \mid u ш v\rangle = 0$).

Proposition 2.2. Let $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$). Then the following assertions are equivalent

i. S is a ж-character (resp. *conc* and ш-character).

ii. S is group-like, for $\Delta_{ж}$ (resp. Δ_{conc} and $\Delta_{ш}$).

iii. $\log S$ is primitive, for $\Delta_{ж}$ (resp. Δ_{conc} and $\Delta_{ш}$).

Corollary 2.1. Let $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle\mathcal{X}\rangle\rangle$). Then the following assertions are equivalent

i. S an infinitesimal ж-character (resp. *conc* and ш-character).

ii. S is primitive, for $\Delta_{ж}$ (resp. Δ_{conc} and $\Delta_{ш}$).

2.2. Factorization in bi-algebras

Due to Cartier-Quillin-Milnor-Moore (Cartier, 1987) theorem (CQMM theorem), it is well known that the enveloping algebra $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\mathcal{X}\rangle)$ is isomorphic to the (connected, graded and co-commutative) bialgebra $\mathcal{H}_{ш}(\mathcal{X}) = (A\langle\mathcal{X}\rangle, conc, 1_{\mathcal{X}^*}, \Delta_{ш}, e)$, where the counit being here $e(P) = \langle P \mid 1\rangle$. Moreover, this algebra is graded and admits a Poincaré-Birkhoff-Witt basis (Reutenauer, 1993) $\{P_w\}_{w\in\mathcal{X}^*}$ which is expanded from the homogeneous basis $\{P_l\}_{l\in \mathrm{Lyn}\mathcal{X}}$ of the Lie algebra of concatenation product, denoted by $\mathcal{L}ie_A\langle\mathcal{X}\rangle$. Its graded dual basis is denoted by $\{S_w\}_{w\in\mathcal{X}^*}$ admitting the pure transcendence basis $\{S_l\}_{l\in\mathrm{Lyn}\mathcal{X}}$ of the A-shuffle algebra.

In the case when A is a $\mathbb{Q}$-algebra, we also have the following factorization of the diagonal series, (Reutenauer, 1993) (here all tensor products are over A)

$$D_{\mathcal{X}} := \sum_{w\in\mathcal{X}^*} w \otimes w = \prod_{l\in\mathrm{Lyn}\mathcal{X}}^{\searrow} e^{S_l \otimes P_l} \qquad (2.5)$$

and (still in the case A is a $\mathbb{Q}$-algebra) dual bases of homogeneous polynomials $\{P_w\}_{w\in\mathcal{X}^*}$ and $\{S_w\}_{w\in\mathcal{X}^*}$ can be constructed recursively as follows

$$\begin{cases} P_x = x, \text{for } x \in \mathcal{X}, \\ P_l = [P_{l_1}, P_{l_2}], (l) = (l_1, l_2), \\ P_w = P_{l_1}^{i_1} \ldots P_{l_k}^{i_k},\ LF(w) = l_1^{i_1} \ldots l_k^{i_k}, \end{cases} \qquad (2.6)$$

where $LF(w)$ denotes the Lyndon factorization of the word w which is rewritten a word as a product of decreasing Lyndon words.

$$\begin{cases} S_x = x,\ x \in \mathcal{X}, \\ S_l = y S_{l'},\ l = yl' \in \mathrm{Lyn}\mathcal{X} - \mathcal{X}, \\ S_w = \dfrac{S_{l_1}^{ш i_1} ш \ldots ш S_{l_k}^{ш i_k}}{i_1! \ldots i_k!},\ LF(w) = l_1^{i_1} \ldots l_k^{i_k}. \end{cases} \qquad (2.7)$$

The graded dual of $\mathcal{H}_{ш}(\mathcal{X})$ is

$$\mathcal{H}_{ш}^{\vee}(\mathcal{X}) = (A\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*}, \Delta_{conc}, \epsilon).$$

We get another connected, graded and co-commutative bialgebra which, in case A is a $\mathbb{Q}$-algebra, is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements,

$$\mathcal{H}_{ж}(Y) = (A\langle Y\rangle, conc, 1_{Y^*}, \Delta_{ж}, \epsilon) \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{ж}(Y))), \qquad (2.8)$$

where

$$\mathrm{Prim}(\mathcal{H}_{\text{ж}}(Y)) = \mathrm{Im}(\pi_1) = \mathrm{span}_A\{\pi_1(w) \mid w \in Y^*\}$$

and π_1 is defined, for any $w \in Y^*$, by (Hoang, 2013b); Bui et al. (2015).

$$\pi_1(w) = w + \sum_{k=2}^{(w)} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k \in Y^+} \langle w | u_1 \text{ж} \ldots \text{ж} u_k \rangle u_1 \ldots u_k . \quad (2.8)$$

Now, let $\{\Pi_w\}_{w\in Y^*}$ be the linear basis, expanded by decreasing Poincaré-Birkhoff-Witt (PBW for short) after any basis $\{\Pi_l\}_{l\in \mathrm{Lyn}Y}$ of $\mathrm{Prim}(\mathcal{H}_{\text{ж}}(Y))$ homogeneous in weight, and let $\{\Sigma_w\}_{w\in Y^*}$ be its dual basis which contains the pure transcendence basis $\{\Sigma_l\}_{l\in \mathrm{Lyn}Y}$ of the A-stuffle algebra. One also has the factorization of the diagonal series D_Y, on $\mathcal{H}_{\text{ж}}(Y)$, which reads (Bui et al., 2013)

$$D_Y := \sum_{w\in Y^*} w \otimes w = \prod_{l\in \mathrm{Lyn}Y}^{\searrow} e^{\Sigma_l \otimes \Pi_l}, \qquad (2.9)$$

where the last expression takes product of exponential in decreasing of Lyndon words.

We are now in the position to state the following

Theorem 2.1 (Hoang, 2013a).

Let A be a $\mathbb{Q}$-algebra, then the endomorphism of algebras

$$\varphi_{\pi_1} : (A\langle Y\rangle, conc, 1_{Y^*}) \to (A\langle Y\rangle, conc, 1_{Y^*})$$

mapping y_k to $\pi_1(y_k)$, is an automorphism of $A\langle Y\rangle$ realizing an isomorphism of bialgebras between $\mathcal{H}_{\text{ш}}(Y)$ and

$$\mathcal{H}_{\text{ж}}(Y) \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\text{ж}}(Y))).$$

In particular, it can be easily checked that the following diagram commutes

$$\begin{array}{ccc} A\langle Y\rangle & \xrightarrow{\Delta_{\text{ш}}} & A\langle Y\rangle \otimes A\langle Y\rangle \\ \downarrow \varphi_{\pi_1} & & \downarrow \varphi_{\pi_1} \otimes \varphi_{\pi_1} \\ A\langle Y\rangle & \xrightarrow{\Delta_{\text{ж}}} & A\langle Y\rangle \otimes A\langle Y\rangle \end{array}$$

Hence, the bases $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ of $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\text{ж}}(Y)))$ are images by φ_{π_1} and by the adjoint mapping of its inverse, $\varphi^{\vee}_{\pi_1}$ of $\{P_w\}_{w\in Y^*}$ and $\{S_w\}_{w\in Y^*}$, respectively. Algorithmically, the dual bases of homogeneous polynomials $\{\Pi_w\}_{w\in Y^*}$ and $\{\Sigma_w\}_{w\in Y^*}$ can be constructed directly and recursively by

$$\begin{cases} \Pi_{y_s} = \pi_1(y_s) \text{for } y_s \in Y, \\ \Pi_l = [\Pi_{l_1}, \Pi_{l_2}] \text{for } l \in \mathrm{Lyn}Y \setminus Y,, \\ \Pi_w = \Pi_{l_1}^{i_1} \ldots \Pi_{l_k}^{i_k} \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, \end{cases} \qquad (2.10)$$

$$\begin{cases} \Sigma_{y_k} = y_k, \\ \Sigma_l = \sum_{(*)_1} y_{s_1} \Sigma_{l_1\ldots l_n} \\ + \sum_{i\geq 2} \frac{1}{i!} \sum_{(*)_2} y_{s'_1+\ldots+s'_i} \Sigma_{l_1\ldots l_n}, \\ \Sigma_w = \dfrac{\Sigma_{l_1}^{\text{ж}\, i_1} \text{ж} \ldots \text{ж}\, \Sigma_{l_k}^{\text{ж}\, i_k}}{i_1! \ldots i_k!}. \end{cases} \qquad (2.11)$$

In $(*)_2$, the sum is taken over all $\{k_1,\ldots,k_i\} \subset \{1,\ldots,k\}$ and $l_1 \geq \ldots \geq l_n$ such that $(y_{s_1},\ldots,y_{s_r}) \overset{*}{\Leftarrow} (y_{s_{k_1}},\ldots,y_{s_{k_i}}, l_1,\ldots,l_n)$, where $\overset{*}{\Leftarrow}$ denotes the transitive closure of the relation on standard sequences, denoted by $\Leftarrow$ (Bui et al., 2013; Reutenauer, 1993).

3. Drinfel'd associator with special functions

3.1. Relations among multiple polylogarithms and multiple harmonic sums

By correspondence (3.1) and the properties of the special functions, we can define the following (morphisms) are injective

$$\mathrm{Li}_{\bullet} : (\mathbb{Q}\langle X\rangle, \text{ш}, 1_{X^*}) \to (\mathbb{Q}\{\mathrm{Li}_w\}_{w\in X^*}, ., 1),$$
$$x_0^n \mapsto \log^n(z)/n!,$$
$$x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1 \mapsto \mathrm{Li}_{x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1}$$

and

$$\mathrm{H}_{\bullet} : (\mathbb{Q}\langle Y\rangle, \text{ш}, 1_{Y^*}) \to (\mathbb{Q}\{\mathrm{H}_w\}_{w\in Y^*}, ., 1),$$
$$y_{s_1} \ldots y_{s_r} \mapsto \mathrm{H}_{y_{s_1}\ldots y_{s_r}} = \mathrm{H}_{s_1,\ldots,s_r}. \qquad (3.2)$$

Hence, the families $\{\mathrm{Li}_w\}_{w\in X^*}$ and $\{\mathrm{H}_w\}_{w\in Y^*}$ are linearly independent.

Now, using D_X and D_Y, the graphs of $\mathrm{Li}_{\bullet}$ and $\mathrm{H}_{\bullet}$ are given as follows (Hoang, 2013b; Bui et al., 2015).

$$\mathrm{L} := (\mathrm{Li}_\bullet \otimes \mathrm{Id})\mathcal{D}_X = \prod_{l \in \mathrm{Lyn}X}^{\searrow} e^{\mathrm{Li}_{S_l} P_l},$$

$$\text{and } \mathrm{L}_{reg} = \prod_{\substack{l \in \mathrm{Lyn}X \\ l \neq x_0, x_1}}^{\searrow} e^{\mathrm{Li}_{S_l} P_l},$$

$$\mathrm{H} := (\mathrm{H}_\bullet \otimes \mathrm{Id})D_Y = \prod_{l \in \mathrm{Lyn}Y}^{\searrow} e^{\mathrm{H}_{\Sigma_l} \Pi_l},$$

$$\text{and } \mathrm{H}_{reg} = \prod_{\substack{l \in \mathrm{Lyn}Y \\ l \neq y_1}}^{\searrow} e^{\mathrm{H}_{\Sigma_l} \Pi_l}.$$

We note that L_{reg} and H_{reg} are generating series in regularization taking convergent words, the words are coded by convergent multi-index of polyzetas. Moreover, we set

$$Z_{ш} := \mathrm{L}_{reg}(1) \text{ and } Z_{ш} := \mathrm{H}_{reg}(+\infty). \qquad (3.3)$$

As for $C_{z_0 \rightsquigarrow z}$, $\mathrm{L}, \mathrm{L}_{reg}$, and then $Z_{ш}$ (resp. $\mathrm{H}, \mathrm{H}_{reg}$, and then $Z_{ш}$) are grouplike, for $\Delta_{ш}$ (resp. $\Delta_{ш}$). Moreover, L is also a solution of (1.4)

Theorem 2.1 (Cristian & Hoang, 2009; Bui et al., 2015)**.**

$$C_{z_0 \rightsquigarrow z} \mathrm{L}(z_0) = \mathrm{L}(z),$$

$$\lim_{z \to 0} \mathrm{L}(z) e^{-x_0 \log(z)} = 1,$$

$$\lim_{z \to 1} e^{x_1 \log(1-z)} \mathrm{L}(z) = Z_{ш}.$$

This means that for $x_0 = A / 2\mathrm{i}\pi$ and $x_1 = -B / 2\mathrm{i}\pi$, L corresponds to G_0 expected by Drindfel'd and $Z_{ш}$ corresponds to Φ_{KZ}, $\mathrm{L}(z)_{z \to 0} e^{x_0 \log(z)}$ and $\mathrm{L}(z)_{z \to 1} e^{-x_1 \log(1-z)} Z_{ш}$. Via Newton-Girard identity type, we also get (Cristian & Hoang, 2009; Bui et al., 2015)

$$\sum_{k \geq 0} \mathrm{H}_{y_1^k}(n) y_1^k = e^{\sum_{k \geq 1} \mathrm{H}_{y_k}(n)(-y_1)^k / k}$$

and then

$$\mathrm{H}(n)_{z \to \infty} \Big(\sum_{k \geq 0} \mathrm{H}_{y_1^k}(n) y_1^k \Big) \pi_Y(Z_{ш}).$$

It follows that

Theorem 2.2 (Cristian & Hoang, 2009; Bui et al., 2015)**.**

$$\pi_Y(Z_{ш}) = \lim_{z \to 1} e^{y_1 \log(1-z)} \pi_Y(\mathrm{L}(z))$$
$$= \lim_{n \to \infty} e^{\sum_{k \geq 1} \mathrm{H}_{y_k}(n)(-y_1)^k / k} \mathrm{H}(n).$$

Hence, the coefficients of any word w in $Z_{ш}$ and $Z_{ж}$ respectively represent the finite parts (denoted by f.p.) of asymptotic expansion of multiple polylogarithm and multiple harmonic sum in the scales of comparison $\{(1-z)^a \log^b((1-z)^{-1})\}_{a \in \mathbb{Z}, b \in \mathbb{N}}, \{n^a \mathrm{H}_1^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$.

This means that

$$\mathrm{f.p.}_{z \to 1} \mathrm{Li}_w(z) = \langle Z_{ш} \mid w \rangle,$$
$$\mathrm{f.p.}_{n \to +\infty} \mathrm{H}_w(n) = \langle Z_{ж} \mid w \rangle.$$

Example 2.1 (Cristian & Hoang, 2009)**.**

In convergence case,

$$\mathrm{Li}_{2,1}(z) = \zeta(3) + (1-z)\log(1-z) - (1-z)^{-1}$$
$$-(1-z)\log^2(1-z)/2$$
$$+(1-z)^2(-\log^2(1-z) + \log(1-z))/4 + \ldots,$$
$$\mathrm{H}_{2,1}(n) = \zeta(3) - (\log(n) + 1 + \gamma)/n + \log(n)/2n + \ldots,$$

one has

$$\mathrm{f.p.}_{z \to 1} \mathrm{Li}_{2,1}(z) = \mathrm{f.p.}_{n \to +\infty} \mathrm{H}_{2,1}(n) = \zeta(2,1) = \zeta(3).$$

In divergence case

$$\mathrm{Li}_{1,2}(z) = 2 - 2\zeta(3) - \zeta(2)\log(1-z)$$
$$-2(1-z)\log(1-z) + (1-z)\log^2(1-z)$$
$$+(1-z)^2(\log^2(1-z) - \log(1-z))/2 + \ldots,$$
$$\mathrm{H}_{1,2}(n) = \zeta(2)\gamma - 2\zeta(3) + \zeta(2)\log(n)$$
$$+(\zeta(2) + 2)/2n + \ldots,$$

since numerically,

$$\zeta(2)\gamma = 0.94948171111498152454556 4\ldots,$$

then one has

$$\mathrm{f.p.}_{z \to 1} \mathrm{Li}_{1,2}(z) = 2 - 2\zeta(3),$$
$$\mathrm{f.p.}_{n \to +\infty} \mathrm{H}_{1,2}(n) = \zeta(2)\gamma - 2\zeta(3).$$

Moreover, the relations among the multiple polylogarithms indexed by basis $\{S_l\}_{l \in \mathrm{Lyn}X}$ follow

$$\mathrm{Li}_{S_{x_0}}(z) = \log(z), \mathrm{Li}_{S_{x_1}}(z) = -\log(1-z),$$
$$\mathrm{Li}_{S_{x_0 x_1}}(z) = -\log(z)\log(1-z) - \mathrm{Li}_{S_{x_0 x_1}}(1-z)$$
$$+\zeta(S_{x_0 x_1}),$$

$$\begin{aligned}\mathrm{Li}_{S_{x_0x_1^2}}(z) &= \frac{1}{2}\log(1-z)^2\log(z)\\ &+\log(1-z)\mathrm{Li}_{S_{x_0x_1}}(1-z)\\ &-\mathrm{Li}_{S_{x_0^2x_1}}(1-z)+\zeta(S_{x_0^2x_1})\\ &+\log(z)\zeta(S_{x_0x_1}).\end{aligned}$$

Using the correspondences given in (3.4), let us consider then the following $\mathbb{Q}$-algebra of convergent polyzetas, being algebraically generated by $\{\zeta(l)\}_{l\in\mathrm{Lyn}X-X}$ (resp. $\{\zeta(S_l)\}_{l\in\mathrm{Lyn}X-X}$), or equivalently, by $\{\zeta(l)\}_{l\in\mathrm{Lyn}Y-\{y_1\}}$ (resp. $\{\zeta(\Sigma_l)\}_{l\in\mathrm{Lyn}Y-\{y_1\}}$):

$$\begin{aligned}\mathcal{Z} &:= \mathrm{span}_{\mathbb{Q}}\{\zeta(w)\}_{w\in x_0X^*x_1}\\ &= \mathrm{span}_{\mathbb{Q}}\{\zeta(w)\}_{w\in Y^*\setminus y_1Y^*}. \qquad (3.5)\end{aligned}$$

For any $k\ge 1$ let

$$\begin{aligned}\mathcal{Z}_k &:= \mathrm{span}_{\mathbb{Q}}\{\zeta(w)\}_{\substack{w\in x_0X^*x_1\\ |w|=k}}\\ &= \mathrm{span}_{\mathbb{Q}}\{\zeta(w)\}_{\substack{w\in(Y-\{y_1\})Y^*\\ (w)=k}}. \qquad (3.6)\end{aligned}$$

Now, considering the third and last noncommutative generating series of polyzetas (Cristian & Hoang, 2009; Bui et al., 2015)

$$Z_\gamma = \sum_{w\in Y^*}\gamma_w w, \qquad (3.7)$$

where $\gamma_w = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n)$ on the scale $\gamma_w = \mathrm{f.p.}_{n\to+\infty}\mathrm{H}_w(n),\{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$.

For any $w\in Y^*\setminus y_1Y^*$, one has $\gamma_w=\zeta(w)$ and $\gamma_{y_1}=\gamma$ (Euler's constant). The series Z_γ is group-like, for $\Delta_{ж}$. Then (Hoang, 2013; Bui Van Chien *et al.*, 2015)

$$Z_\gamma = e^{\gamma y_1}\prod_{l\in\mathrm{Lyn}Y\setminus\{y_1\}}^{\searrow} e^{\zeta(l)\Pi_l} = e^{\gamma y_1}Z_{ш}. \qquad (3.8)$$

Moreover, introducing the following ordinary generating series[i]

$$B(y_1) := \exp\Big(\gamma y_1 - \sum_{k\ge 2}\zeta(k)\frac{(-y_1)^k}{k}\Big), \qquad (3.9)$$

$$B'(y_1) := \exp\Big(\sum_{k\ge 2}\zeta(k)\frac{(-y_1)^k}{k}\Big), \qquad (3.10)$$

we obtain the following bridge equation

Theorem 2.3 (Hoang, 2013b; Bui et al., 2015)**.**

$$Z_\gamma = B(y_1)\pi_Y Z_{ш} \qquad (3.11)$$

or equivalently by simplification

$$Z_{ж} = B'(y_1)\pi_Y Z_{ш} \qquad (3.12)$$

Identifying the coefficients in these identities, we get

$$\gamma_{y_1^k} = \sum_{s_1,\dots,s_r\ge 1, s_1+\dots+ks_r=k}\frac{(-1)^k}{s_1!\dots s_r!}(-\gamma)^{s_1}\Big(-\frac{\zeta(2)}{2}\Big)^{s_2}\dots\Big(-\frac{\zeta(k)}{k}\Big)^{s_r},$$

$$\gamma_{y_1^k w} = \sum_{i=0}^{k}\frac{\zeta(x_0[(-x_1)^{k-i}ш\pi_X w])}{i!}\Big(\sum_{j=1}^{i}b_{i,j}(\gamma,-\zeta(2),2\zeta(3),\dots)\Big),$$

where $k\in\mathbb{N}_+, w\in Y^+$ and $b_{n,k}(t_1,\dots,t_k)$ are Bell polynomials.

Example 2.2 (Cristian and Hoang, 2009)**.**

With the correspondences given in (3.13), we get

$$\begin{aligned}\gamma_{1,1} &= \frac{1}{2}(\gamma^2-\zeta(2)),\gamma_{1,1,1}\\ &= \frac{1}{6}(\gamma^3-3\zeta(2)\gamma+2\zeta(3)).\end{aligned}$$

$$\gamma_{1,7} = \zeta(7)\gamma+\zeta(3)\zeta(5)-\frac{54}{175}\zeta(2)^4,$$

$$\begin{aligned}\gamma_{1,1,6} &= \frac{4}{35}\zeta(2)^3\gamma^2+(\zeta(2)\zeta(5)+\frac{2}{5}\zeta(3)\zeta(2)^2\\ &-4\zeta(7))\gamma+\zeta(6,2)+\frac{19}{35}\zeta(2)^4\\ &+\frac{1}{2}\zeta(2)\zeta(3)^2-4\zeta(3)\zeta(5).\end{aligned}$$

3.2. Relations of polyzetas

As the limits $\lim_{z\to 1}\mathrm{Li}_s(z)=\lim_{n\to\infty}\mathrm{H}_s(n)=\zeta(s)$ for any convergent multi-index[ii] s, polyzetas inherits properties both of multiple polylogarithms and multiple harmonic sums. We can define polyzetas as a morphism of shuffle and quasi-shuffle products from $(\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}Xx_1, ш, 1_{X^*})$ or $(\mathbb{Q}1_{Y^*}\oplus(Y\setminus\{y_1\})\mathbb{Q}\langle Y\rangle, ж, 1_{Y^*})$ onto $\mathbb{Q}$-algebra, denoted by $\mathcal{Z}$, algebraically generated by the convergent polyzetas

$\{\zeta(l)\}_{l\in\mathrm{Lyn}X-X}$ (Bui et al., 2015). It can be extended as characters

$$\zeta\ :(\mathbb{Q}\langle X\rangle,,1_{X^*})\to(\mathbb{R},.,1),$$

$$\zeta,\gamma_{\cdot}:(\mathbb{Q}\langle Y\rangle,\text{ж},1_{Y^*})\to(\mathbb{R},.,1)$$

such that, for any $w\in X^*$, one has the finite part corresponding the scales $\{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, $\{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ and $\{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$ as follows

$$\zeta_{\text{ш}}(w)=\text{f.p.}_{z\to1}\mathrm{Li}_w(z),$$
$$\zeta_{\text{ж}}(\pi_Y w)=\text{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y w}(n),$$
$$\gamma_{\pi_Y w}=\text{f.p.}_{n\to+\infty}\mathrm{H}_{\pi_Y l}(n).$$

It follows that, $\zeta_{\text{ш}}(x_0)=0=\log(1)$ and the finite parts, corresponding the scales $\{(1-z)^a\log^b(1-z)\}_{a\in\mathbb{Z},b\in\mathbb{N}},\{n^a\mathrm{H}_1^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, $\{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$, as follows

$$\zeta_{\text{ш}}(x_1)=0=\text{f.p.}_{z\to1}\log(1-z),$$
$$\zeta_{\text{ш}}(y_1)=0=\text{f.p.}_{n\to+\infty}\mathrm{H}_1(n),$$
$$\gamma_{y_1}=\gamma=\text{f.p.}_{n\to+\infty}\mathrm{H}_1(n)$$

and the following convergent polyzetas,
$\forall l\in\mathrm{Lyn}X-X$,

$$\zeta_{\text{ш}}(l)=\zeta_{\text{ж}}(\pi_Y l)=\gamma_{\pi_Y l}=\zeta(l),$$
$$\zeta_{\text{ш}}(S_l)=\zeta_{\text{ж}}(\pi_Y S_l)=\gamma_{\pi_Y S_l}=\zeta(S_l),$$

$\forall l\in\mathrm{Lyn}Y-\{y_1\}$,

$$\zeta_{\text{ж}}(l)=\zeta_{\text{ш}}(\pi_X l)=\gamma_l=\zeta(l),$$
$$\zeta_{\text{ж}}(\Sigma_l)=\zeta_{\text{ш}}(\pi_X\Sigma_l)=\gamma_{\Sigma_l}=\zeta(\Sigma_l).$$

In (Cristian & Hoang, 2009), polynomial relations among $\{\zeta(l)\}_{l\in\mathrm{Lyn}X-X}$ (or $\{\zeta(l)\}_{l\in\mathrm{Lyn}Y-\{y_1\}}$), are obtained using the double shuffle relations. The identification of local coordinates in $Z_\gamma=B(y_1)\pi_Y Z_{\text{ш}}$, leads to a family of algebraic generators $\mathcal{Z}_{irr}^{\infty}(X)$ of $\mathcal{Z}$

$$\mathcal{Z}_{irr}^{\le2}(\mathcal{X})\subset\cdots\subset\mathcal{Z}_{irr}^{\le p}(\mathcal{X})\subset\cdots\subset\mathcal{Z}_{irr}^{\infty}(\mathcal{X})=\bigcup_{p\ge2}\mathcal{Z}_{irr}^{\le p}(\mathcal{X})$$

and their inverse image by a section of ζ

$$\mathcal{L}_{irr}^{\le2}(\mathcal{X})\subset\cdots\subset\mathcal{L}_{irr}^{\le p}(\mathcal{X})\subset\cdots\subset\mathcal{L}_{irr}^{\infty}(\mathcal{X})=\bigcup_{p\ge2}\mathcal{L}_{irr}^{\le p}(\mathcal{X})$$

such that the following restriction is bijective

$$\zeta:\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)]\to\mathcal{Z}=\mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]=\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}_{irr}^{\infty}(\mathcal{X})}].$$

Moreover, the following sub ideals of $\ker\zeta$

$$R_Y:=(\mathrm{span}_{\mathbb{Q}}\{Q_l\}_{l\in\mathrm{Lyn}Y\setminus\{y_1\}},\text{ж},1_{Y^*}),$$
$$R_X:=(\mathrm{span}_{\mathbb{Q}}\{Q_l\}_{l\in\mathrm{Lyn}X\setminus X},\text{ш},1_{X^*})$$

are generated by the polynomials $\{Q_l\}_{l\in\mathrm{Lyn}\mathcal{X},\ l\notin\{y_1,x_0,x_1\}}$ homogeneous in weight such that the following assertions are equivalent:

i. $Q_l=0$,

ii. $\Sigma_l\to\Sigma_l$ (resp. $S_l\to S_l$),

iii. $\Sigma_l\in\mathcal{L}_{irr}^{\infty}(Y)$ (resp. $S_l\in\mathcal{L}_{irr}^{\infty}(X)$).

Any polynomial Q_l ($\neq0$) is led by Σ_l (resp. S_l), being transcendent over the sub algebra $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, and $\Sigma_l\to\Upsilon_l$ (resp. $S_l\to U_l$) being homogeneous of weight $p=(l)$ and belonging to

$\mathbb{Q}[\mathcal{L}_{irr}^{\le p}(\mathcal{X}))]$. In other terms, $\Sigma_l=Q_l+\Upsilon_l$ i.e. $\mathrm{span}_{\mathbb{Q}}\{S_l\}_{l\in\mathrm{Lyn}X\setminus X}=R_{\mathcal{X}}\oplus\mathrm{span}_{\mathbb{Q}}\mathcal{L}_{irr}^{\infty}(\mathcal{X})$ (resp. $S_l=Q_l+U_l$ which follows $\mathrm{span}_{\mathbb{Q}}\{\Sigma_l\}_{l\in\mathrm{Lyn}Y\setminus\{y_1\}}=R_{\mathcal{X}}\oplus\mathrm{span}_{\mathbb{Q}}\mathcal{L}_{irr}^{\infty}(\mathcal{X})$.

For any $w\in x_0X^*x_1$ (resp. $Y\setminus\{y_1\})Y^*$), by the Radford's theorem (Reutenauer, 1993), one has $\zeta(w)\in\mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]$. Hence, for any $P\in\mathbb{Q}[\{S_l\}_{l\in\mathrm{Lyn}X\setminus X}]$ (resp. $\mathbb{Q}[\{\Sigma_l\}_{l\in\mathrm{Lyn}Y\setminus\{y_1\}}]$ such that $P\notin\ker\zeta\supseteq R_{\mathcal{X}}$, one gets, by linearity, $\zeta(P)\in\mathbb{Q}[\mathcal{Z}_{irr}^{\infty}(\mathcal{X})]$.

Next, let $Q\in R_{\mathcal{X}}\cap\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$. Since $R_{\mathcal{X}}\subseteq\ker\zeta$ then $\zeta(Q)=0$. Moreover, restricted on $\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(\mathcal{X})]$, the polymorphism ζ is bijective and then $Q=0$. It follows that

Proposition 2.3 (Hoang, 2013b; Bui et al., 2015).

$$\mathbb{Q}[\{S_l\}_{l\in\mathrm{Lyn}X\setminus X}]=R_X\oplus\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(X)],$$
$$\mathbb{Q}[\{\Sigma_l\}_{l\in\mathrm{Lyn}Y\setminus\{y_1\}}]=R_Y\oplus\mathbb{Q}[\mathcal{L}_{irr}^{\infty}(Y)].$$

Via CQMM theorem and by duality, one deduces then

Corollary 2.2.

$$\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle \setminus X) = \mathcal{J}_X \oplus \mathcal{U}\left(\mathcal{L}ie_{\mathbb{Q}}\left\langle \{P_l\}_{\substack{l\in\mathrm{Lyn}X \\ S_l\in\mathcal{L}^{\infty}_{irr}(X)}} \right\rangle\right),$$

$$\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle \setminus \{y_1\}) = \mathcal{J}_Y \oplus \mathcal{U}\left(\mathcal{L}ie_{\mathbb{Q}}\left\langle \{\Pi_l\}_{\substack{l\in\mathrm{Lyn}X \\ \Sigma_l\in\mathcal{L}^{\infty}_{irr}(Y)}} \right\rangle\right),$$

where $\mathcal{J}_X$ (resp. $\mathcal{J}_Y$) is a Lie ideal generated by $\{P_l\}_{l\in\mathrm{Lyn}X:S_l\notin\mathcal{L}^{\infty}_{irr}(X)}$ (resp. $\{\Pi_l\}_{l\in\mathrm{Lyn}Y:\Sigma_l\notin\mathcal{L}^{\infty}_{irr}(Y)}$).

Now, let $Q\in\ker\zeta$, $\langle Q 1_{\mathcal{X}^*}=0$. Then $Q=Q_1+Q_2$ with $Q_1\in R_{\mathcal{X}}$ and $Q_2\in\mathbb{Q}[\mathcal{L}^{\infty}_{irr}(\mathcal{X})]$. Thus, $Q\equiv_{R_{\mathcal{X}}} Q_1\in R_{\mathcal{X}}$ and then

Corollary 2.3 (Hoang, 2013b; Bui et al., 2015)**.**

$$\mathbb{Q}[\{\zeta(p)\}_{p\in\mathcal{L}^{\infty}_{irr}}(\mathcal{X})]=\mathcal{Z}=\mathrm{Im}\zeta \text{ and } R_{\mathcal{X}}=\ker\zeta.$$

On the other hand, one also has

$$\mathcal{Z}\cong\mathbb{Q}1_{Y^*}\oplus(Y-\{y_1\})\mathbb{Q}\langle Y\rangle/\ker\zeta$$
$$\cong\mathbb{Q}1_{X^*}\oplus x_0\mathbb{Q}Xx_1/\ker\zeta.$$

Hence, as an ideal generated by homogeneous in weight polynomials, $\ker\zeta$ is graded and so is $\mathcal{Z}$:

Corollary 2.4 (Hoang, 2013b; Bui et al., 2015)**.**

$$\mathcal{Z}=\mathbb{Q}1\oplus\bigoplus_{k\geq 2}\mathcal{Z}_k. \tag{3.14}$$

Now, let $\xi=\zeta(P)$, where $P\in\mathbb{Q}\langle\mathcal{X}\rangle$ and $P\notin\ker\zeta$, homogeneous in weight. Since, for any p and $n\geq 1$, one has $\mathcal{Z}_p\mathcal{Z}_n\subset\mathcal{Z}_{p+n}$ then each monomial ξ^n, for $n\geq 1$, is of different weight. Thus ξ could not satisfy

$$\xi^n+a_{n-1}\xi^{n-1}+\ldots=0, \text{with } a_{n-1},\ldots\in\mathbb{Q}.$$

Corollary 2.5 (Hoang, 2013b; Bui et al., 2015). Any $s\in\mathcal{L}^{\infty}_{irr}(\mathcal{X})$ is homogeneous in weight then $\zeta(s)$ is transcendent over $\mathbb{Q}$.

Example 2.3 Polynomials relations on local coordinates (Bui et al., 2015)**.** Due to the bridge equation (3.12), we obtain Table 1.

Table 1. Polynomial relations of polyzetas on transcendence bases

	Relations on $\{\zeta(\Sigma_l)\}_{l\in\mathcal{L}ynY-\{y_1\}}$	Relations on $\{\zeta(S_l)\}_{l\in\mathcal{L}ynX-X}$
3	$\zeta(\Sigma_{y_2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})$	$\zeta(S_{x_0x_1^2}) = \zeta(S_{x_0^2x_1})$
4	$\zeta(\Sigma_{y_4}) = \frac{2}{5}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^3x_1}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_3y_1}) = \frac{3}{10}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0^2x_1^2}) = \frac{1}{10}\zeta(S_{x_0x_1})^2$
	$\zeta(\Sigma_{y_2y_1^2}) = \frac{2}{3}\zeta(\Sigma_{y_2})^2$	$\zeta(S_{x_0x_1^3}) = \frac{2}{5}\zeta(S_{x_0x_1})^2$
5	$\zeta(\Sigma_{y_3y_2}) = 3\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - 5\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^3x_1^2}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_4y_1}) = -\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{2}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1x_0x_1}) = -\frac{3}{2}\zeta(S_{x_0^4x_1}) + \zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1})$
	$\zeta(\Sigma_{y_2^2y_1}) = \frac{3}{2}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) - \frac{25}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0^2x_1^3}) = -\zeta(S_{x_0^2x_1})\zeta(S_{x_0x_1}) + 2\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_3y_1^2}) = \frac{5}{12}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1x_0x_1^2}) = \frac{1}{2}\zeta(S_{x_0^4x_1})$
	$\zeta(\Sigma_{y_2y_1^3}) = \frac{1}{4}\zeta(\Sigma_{y_3})\zeta(\Sigma_{y_2}) + \frac{5}{4}\zeta(\Sigma_{y_5})$	$\zeta(S_{x_0x_1^4}) = \zeta(S_{x_0^4x_1})$
6	$\zeta(\Sigma_{y_6}) = \frac{8}{35}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^5x_1}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_2}) = \zeta(\Sigma_{y_3})^2 - \frac{4}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^4x_1^2}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_5y_1}) = \frac{2}{7}\zeta(\Sigma_{y_2})^3 - \frac{1}{2}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1x_0x_1}) = \frac{4}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_3y_1y_2}) = -\frac{17}{30}\zeta(\Sigma_{y_2})^3 + \frac{9}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^3x_1^3}) = \frac{23}{70}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_2y_1}) = 3\zeta(\Sigma_{y_3})^2 - \frac{9}{10}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0^2x_1x_0x_1^2}) = \frac{2}{105}\zeta(S_{x_0x_1})^3$
	$\zeta(\Sigma_{y_4y_1^2}) = \frac{3}{10}\zeta(\Sigma_{y_2})^3 - \frac{3}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^2x_0x_1}) = -\frac{89}{210}\zeta(S_{x_0x_1})^3 + \frac{3}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2^2y_1^2}) = \frac{11}{63}\zeta(\Sigma_{y_2})^3 - \frac{1}{4}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0^2x_1^4}) = \frac{6}{35}\zeta(S_{x_0x_1})^3 - \frac{1}{2}\zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_3y_1^3}) = \frac{1}{21}\zeta(\Sigma_{y_2})^3$	$\zeta(S_{x_0x_1x_0x_1^3}) = \frac{8}{21}\zeta(S_{x_0x_1})^3 - \zeta(S_{x_0^2x_1})^2$
	$\zeta(\Sigma_{y_2y_1^4}) = \frac{17}{50}\zeta(\Sigma_{y_2})^3 + \frac{3}{16}\zeta(\Sigma_{y_3})^2$	$\zeta(S_{x_0x_1^5}) = \frac{8}{35}\zeta(S_{x_0x_1})^3$

Example 2.4 (Bui et al. 2015). List of irreducible polyzetas up to weight 12 for each transcendence basis:

$$\mathcal{Z}_{irr}^{\leq 12}(X)=\{\zeta(S_{x_0x_1}),\ \zeta(S_{x_0^2x_1}),\ \zeta(S_{x_0^4x_1}),\ \zeta(S_{x_0^6x_1}),\ \zeta(S_{x_0x_1^2x_0x_1^4}),\ \zeta(S_{x_0^8x_1}),\ \zeta(S_{x_0x_1^2x_0x_1^6}),\ \zeta(S_{x_0^{10}x_1}),\ \zeta(S_{x_0x_1^3x_0x_1^7}),\ \zeta(S_{x_0x_1^2x_0x_1^8}),\ \zeta(S_{x_0x_1^4x_0x_1^6})\}.$$

$$\mathcal{Z}_{irr}^{\leq 12}(Y)=\{\zeta(\Sigma_{y_2}),\ \zeta(\Sigma_{y_3}),\ \zeta(\Sigma_{y_5}),\ \zeta(\Sigma_{y_7}),\ \zeta(\Sigma_{y_3y_1^5}),\ \zeta(\Sigma_{y_9}),\ \zeta(\Sigma_{y_3y_1^7}),\ \zeta(\Sigma_{y_{11}}),\ \zeta(\Sigma_{y_2y_1^9}),\ \zeta(\Sigma_{y_3y_1^9}),\ \zeta(\Sigma_{y_2^2y_1^8})\}.$$

4. Conclusion

We reviewed a method to reduce relations of the special functions indexed by transcendence bases of shuffle and quasi-shuffle algebras due to the Drinfel'd associator. Starting from the research of Knizhnik-Zamolodchikov about a form of a differential equation, a bridge equation is constructed, and it can be applied to the case of the generating series of the special functions. Relations in form of asymptotic expansions or explicit representations hold by the identification of local coordinates of the bridge equation.

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[i] For any $k\geq 1, \langle \log B(y_1) \mid y_1^k\rangle = \text{f.p.}_{n\to+\infty}\left\langle \sum_{l\geq 1} \mathrm{H}_{y_l}(n)(-y_1)^l / l \,\middle|\, y_1^k \right\rangle, \{n^a \log^b(n)\}_{a\in\mathbb{Z}, b\in\mathbb{N}}$.

[ii] $s=(s_1,\ldots,s_r)\in\mathbb{N}^r$ is a convergent multi-index if $s_1\geq 2$.