SYNCHRONIZATION IN COMPLETE NETWORKS OF ORDINARY DIFFERENTIAL EQUATIONS OF FITZHUGH-NAGUMO TYPE WITH NONLINEAR COUPLING

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Abstract

Synchronization is a ubiquitous feature in many natural systems and nonlinear science. This paper studies the synchronization in a complete network consisting of n nodes. Each node is connected to all other nodes by nonlinear coupling and represented by an ordinary differential system of FitzHugh-Nagumo type (FHN) which can be obtained by simplifying the famous Hodgkin-Huxley model. From this complete network, a sufficient condition on the coupling strength is identified to achieve the synchronization. The result shows that the networks with bigger in-degrees of the nodes synchronize more easily. The paper also shows this theoretical result numerically and see that there is a compromise.

Keywords: *Coupling strength, complete network, FitzHugh-Nagumo model, nonlinear coupling, synchronization.*

SỰ CỘNG HƯỞNG TRONG MẠNG LƯỚI ĐẦY ĐỦ CÁC PHƯƠNG TRÌNH VI PHÂN DẠNG FITZHUGH – NAGUMO VỚI LIÊN KẾT PHI TUYẾN

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Tóm tắt

Sự cộng hưởng là một tính năng phổ biến trong nhiều hệ thống tự nhiên và khoa học phi tuyến. Bài báo này nghiên cứu về sự cộng hưởng trong mạng lưới đầy đủ bao gồm n nút. Trong đó, mỗi nút được liên kết với tất cả các nút khác bằng liên kết phi tuyến tính và mỗi nút sẽ được giới thiệu bằng một hệ phương trình vi phân dạng FitzHugh-Nagumo (FHN), đây là một mô hình đơn giản hóa từ mô hình nổi tiếng Hodgkin-Huxley. Từ mạng lưới đầy đủ này, chúng tôi tìm điều kiện đủ cho độ mạnh liên kết để có được sự cộng hưởng. Kết quả cho thấy rằng mạng lưới có các nút mà liên kết đầu vào càng lớn thì cộng hưởng càng dễ. Bài báo còn đưa ra kết quả kiểm tra phương pháp lý thuyết này bằng phương pháp số và xét sự tương quan của hai phương pháp.

Từ khóa: *Độ mạnh liên kết, mạng lưới đầy đủ, mô hình FitzHugh-Nagumo, liên kết phi tuyến, sự cộng hưởng.*

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1. Introduction

Synchronization is a ubiquitous feature in many natural systems and nonlinear science. The word "*synchronization*" is of Greek origin, with *syn* as "common" and *chronous* as "time", which means having the same behavior at the same time. Therefore, the synchronization of two dynamical systems usually means that one system copies the movement of the other. When the behaviors of many systems are synchronized, these systems are called *synchronous*. Studies by Aziz-Alaoui (2006) & Corson (2009) suggested that a phenomenon of synchronization may appear in a network of many weakly coupled oscillators. A broad variety of applications have emerged to increase the power of lasers, synchronize the output of electric circuits, control oscillations in chemical reactions or encode electronic messages for secure communications (Pikovsky et al., 2001; Aziz-Alaoui, 2006).

Synchronization has been extensively studied in many fields and many natural phenomena reflect the synchronization such as the movement of birds forming the cloud, the movement of fishes in the lake, the movement of the parade, the reception and transmission of a group of cells (Hodgkin & Huxley, 1952; Murray, 2002; Izhikevich, 2005; Aziz-Alaoui, 2006; Ermentrout & Terman, 2009). Therefore, the study of the synchronization in the network of cells is very necessary. In order to make the study easier, a complete network of n neurons interconnected together with non-linear coupling is investigated and the sufficient condition on the coupling strength is sought to achieve the synchronization. Each neuron is represented by a dynamical system named FitzHugh-Nagumo model. It was introduced as a dimensional reduction of the well-known Hodgkin-Huxley model (Hodgkin, 1952; Nagumo, 1962; Murray, 2002; Izhikevich, 2007; Ermentrout, 2009; Keener, 2009). It is more analytically tractable and maintains some biophysical meaning. The model is constituted a common form of two equations in the two variables u and v. The first variable is the fast one called excitatory representing the transmembrane voltage. The second one is the slow recovery variable describing the time dependence of several physical quantities, such as electrical conductivity of ion currents across the membrane. The FitzHugh-Nagumo equations (FHN) are given by:

$$\begin{cases} \varepsilon \dfrac{du}{dt} = f(u) - v \\ \dfrac{dv}{dt} = au - bv + c \end{cases} \tag{1}$$

where a, b and c are constants (a and b are positive), $0 < \varepsilon < 1$, $t \geq 0$ and

$$f(u) = -u^3 + 3u.$$

The system (1) is considered as a neural model and from this, a network of n coupled systems (1) based on FHN type is constructed as follows:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i - h(u_i, u_j) \\ v_{it} = au_i - bv_i + c \\ i, j = 1, \ldots, n, i \neq j, \end{cases} \tag{2}$$

where $(u_i, v_i), i = 1, 2, \ldots, n$ is defined by (1).

The function h is the coupling function that determines the type of connection between neurons u_i and u_j. Connections between neurons are essentially of two types: chemical connection and electical connection, where chemical connection is more abundant than electrical one. If the connections are made by chemical synapse, the coupling is non-linear and given by the function:

$$h(u_i, v_i) = (u_i - V_{syn}) g_{syn} \sum_{j=1}^{n} c_{ij} \Gamma(u_j), \tag{3}$$
$$i = 1, 2, \ldots, n.$$

The parameter g_{syn} represents the coupling strength. The coefficients c_{ij} are the elements of the connectivity matrix

$C_n = (c_{ij})_{n\times n}$, defined by: $c_{ij} = 1$ if u_i and u_j are coupled, $c_{ij} = 0$ if u_i and u_j are not coupled, where $i, j = 1, 2, ..., n, i \neq j$.

The function Γ is a non-linear threshold function:

$$\Gamma(u_j) = \frac{1}{1+\exp(-\lambda(u_j - \theta_{syn}))}, \quad j = 1, 2, ..., n.$$

The parameters have the following physiological meanings:

- V_{syn} is the reversal potential and must be larger than u_i, for all $i = 1, 2, ...n, t \geq 0$ since synapses are supposed excitatory.
- θ_{syn} is the threshold reached by every action potential for a neuron.
- λ is a positive number (Belykh et al., 2005; Corson, 2009). The bigger λ is, the better we approach the Heaviside function.

In recent years, there are a lot of studies on the synchronization (Ambrosio & Aziz-Alaoui, 2012; Ambrosio & Aziz-Alaoui, 2013; Corson, 2009); however, they are just studied for the linear coupling, while the connections between neurons made by chemical synapse is major in neural networks. It means that the coupling is nonlinear. Therefore, it is really useful to conduct research on this problem. In other words, we are interested in the rapid chemical excitatory synapses, so the parameters are fixed as follows throughout this paper, based on previous reports (Belykh et al., 2005; Corson, 2009).

$$\lambda = 10, V_{syn} = 2, \theta_{syn} = -0,25.$$

2. Synchronization of a complete network

In this paper, the synchronization is investigated in a complete network, i.e. each node connects to all other nodes of the network (Ambrosio & Aziz-Alaoui, 2012; Ambrosio & Aziz-Alaoui, 2013). For example, Figure 1 shows the complete graphs from 3 to 10 nodes. Each node represents a neuron modeled by a dynamical system of FHN type and each edge represents a synaptic connection modeled by a nonlinear coupling function. A network of n "neurons" (1) bi-directionally coupled by the chemical synapses, based on FHN, is given as follows:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i - \sum\limits_{k=1,k\neq i}^{n} \dfrac{g_n(u_i - V_{syn})}{1+\exp(-\lambda(u_k - \theta_{syn}))} \\ v_{it} = au_i - bv_i + c \\ i = 1, 2, ..., n, \end{cases} \tag{4}$$

where a, b and c are constants (a and b are positive), $0 < \varepsilon < 1, t \geq 0$ and $f(u) = -u^3, +3u$, g_n is the coupling strength between u_i and u_j.

Definition 1 (Aziz-Alaoui, 2006**).** *Let* $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ *and* $S = (S_1, S_2, ..., S_n)$ *be a network. We say that* S *is synchronous if*

$$\lim_{t\to+\infty} |u_j - u_i| = 0 \text{ and } \lim_{t\to+\infty} |v_j - v_i| = 0,$$
$$i, j = 1, 2, ..., n.$$

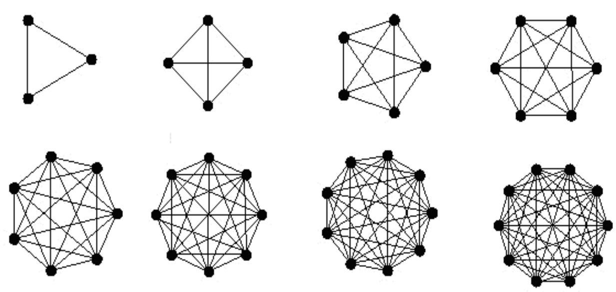

Figure 1. Complete graphs from 3 to 10 nodes. In this study, each node represents a neuron modeled by a dynamical system of FHN type and each edge represents a synaptic connection modeled by a nonlinear coupling function

Theorem 1. *Let*

$$N = \inf\{u_i(t), i = 1, 2, ..., n, t \geq 0\}$$

and suppose that

$$g_n > \frac{M\left[1+\exp(-\lambda(N-\theta_{syn}))\right]}{n-1}, \qquad (5)$$

where $M = \sup_{u\in B, x\in\mathbb{R}} \sum_{k=1}^{3} \frac{f^{(k)}(u)}{k!} x^{k-1}$, *B is a compact interval including u and* $f^{(k)}(u)$ *is the* kth *derivative of f with respect to* u. *Then the network (4) synchronizes in the sense of Definition 1.*

Remark 1. The existence of *B* was proved in Ambrosio et al. (2018). Since the variables *u(t), v(t)* of FHN are bounded (Ambrosio, 2009), $N = \inf\{u_i(t), i=1,2,...,n, t\geq 0\}$ exists.

Proof. Let

$$\Phi(t) = \frac{1}{2}\left[\sum_{i=2}^{n}\left(a\varepsilon(u_i-u_1)^2+(v_i-v_1)^2\right)\right].$$

By deriving the function $\Phi(t)$ with respect to t, we have:

$$\begin{aligned}
\frac{d\Phi(t)}{dt} &= \sum_{i=2}^{n}\left[a\varepsilon(u_i-u_1)(u_{it}-u_{1t})+\right.\\
&\qquad \left.(v_i-v_1)(v_{it}-v_{1t})\right]\\
&= \sum_{i=2}^{n}\left[a(u_i-u_1)\left(f(u_i)-v_i-\right.\right.\\
&\quad \sum_{k=1,k\neq i}^{n}\frac{g_n(u_i-V_{syn})}{1+\exp(-\lambda(u_k-\theta_{syn}))}-f(u_1)+v_1+\\
&\quad \left.\sum_{l=2}^{n}\frac{g_n(u_1-V_{syn})}{1+\exp(-\lambda(u_l-\theta_{syn}))}\right)+\\
&\quad \left.(v_i-v_1)\left(a(u_i-u_1)-b(v_i-v_1)\right)\right]\\
&= \sum_{i=2}^{n}\left[a(u_i-u_1)\left(f(u_i)-f(u_1)-\right.\right.\\
&\quad \sum_{k=1,k\neq i}^{n}\frac{g_n(u_i-V_{syn})}{1+\exp(-\lambda(u_k-\theta_{syn}))}\\
&\quad \left.\left.+\sum_{l=2}^{n}\frac{g_n(u_1-V_{syn})}{1+\exp(-\lambda(u_l-\theta_{syn}))}\right)-b(v_i-v_1)^2\right]\\
&\leq \sum_{i=2}^{n}\left[a(u_i-u_1)\left(f(u_i)-f(u_1)-\right.\right.\\
&\quad \sum_{k=1,k\neq i}^{n}\frac{g_n(u_i-u_1)}{1+\exp(-\lambda(u_k-\theta_{syn}))}+\\
&\quad g_n(u_1-V_{syn})\left(\sum_{l=2}^{n}\frac{1}{1+\exp(-\lambda(u_l-\theta_{syn}))}-\right.\\
&\quad \left.\left.\left.\sum_{k=1,k\neq i}^{n}\frac{1}{1+\exp(-\lambda(u_k-\theta_{syn}))}\right)-b(v_i-v_1)^2\right]\right.\\
&\leq \sum_{i=2}^{n}\left[a(u_i-u_1)\left(f(u_i)-f(u_1)-\right.\right.\\
&\quad \sum_{k=1,k\neq i}^{n}\frac{g_n(u_i-u_1)}{1+\exp(-\lambda(u_k-\theta_{syn}))}\\
&\quad +g_n(u_1-V_{syn})\left(\frac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))}-\right.\\
&\quad \left.\left.\left.\frac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\right)-b(v_i-v_1)^2\right]\right..
\end{aligned}$$

Since we are interested in the rapid chemical excitatory synapses, $u_1 < V_{syn}, \forall t\geq 0 \Rightarrow u_1 - V_{syn} < 0, \forall t \geq 0.$

Note that :

- If $u_i > u_1$, then

$$u_i - u_1 > 0 \Rightarrow g_n(u_i-u_1)(u_1-V_{syn}) < 0,$$

and

$$\frac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))} > \frac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}.$$

Thus

$$g_n(u_i-u_1)(u_1-V_{syn})\left(\frac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))}-\frac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\right)<0.$$

- If $u_i < u_1$, then

$$u_i - u_1 < 0 \Rightarrow g_n(u_i-u_1)(u_1-V_{syn}) > 0,$$

and

$$\frac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))} < \frac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}.$$

Thus

$$g_n(u_i-u_1)(u_1-V_{syn})\left(\frac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))}-\frac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\right)<0.$$

It means that in any cases, there is always the inequality:

$$g_n(u_i-u_1)(u_1-V_{syn})\left(\frac{1}{1+\exp(-\lambda(u_i-\theta_{syn}))}-\frac{1}{1+\exp(-\lambda(u_1-\theta_{syn}))}\right)<0.$$

Therefore,

$$\frac{d\Phi(t)}{dt}\le\sum_{i=2}^{n}\left[a(u_i-u_1)^2\left(f'(u_1)+\sum_{k=2}^{3}\frac{f^{(k)}(u_1)}{k!}(u_i-u_1)^{k-1}-\sum_{k=1,k\neq i}^{n}\frac{g_n}{1+\exp(-\lambda(u_k-\theta_{syn}))}\right)-b(v_i-v_1)^2\right]$$

$$\le\sum_{i=2}^{n}\left[a(u_i-u_1)^2\left(M-\sum_{k=1,k\neq i}^{n}\frac{g_n}{1+\exp(-\lambda(u_k-\theta_{syn}))}\right)-b(v_i-v_1)^2\right].$$

Since $g_n>\dfrac{M\left[1+\exp(-\lambda(N-\theta_{syn}))\right]}{n-1}$, then

$$M-\sum_{k=1,k\neq i}^{n}\frac{g_n}{1+\exp(-\lambda(u_k-\theta_{syn}))}\le M-\frac{(n-1)g_n}{1+\exp(-\lambda(N-\theta_{syn}))}<0.$$

Finally, there is always another constant $\beta>0$, such that

$$\frac{d\Phi(t)}{dt}\le-\beta\Phi(t)\Rightarrow\Phi(t)\le\Phi(0)e^{-\beta t},$$

where

$$\beta=\min\left(\frac{2}{\varepsilon}\left[\frac{(n-1)g_n}{1+\exp(-\lambda(N-\theta_{syn}))}-M\right],2b\right).$$

Thus, there is the synchronization if the coupling strength is verified (5).

3. Numerical simulations

This research focuses on the minimal values of coupling strength g_n to observe a phenomenon of synchronization between n subsystems modeling the function of neuron network.

In the following, the paper shows the numerical results obtained by integrating the system (4) where $n=2, f(u)=-u^3+3u$, with the following parameter values: $a=1; b=0.001; c=0; \varepsilon=0.1; \lambda=10; V_{syn}=2;$ $\theta_{syn}=-0.25.$ The integration of system is realized by using C++ and the results are represented by Gnuplot.

Figure 2 illustrates the synchronization of the complete network of 2 neurons. The simulations show that the system synchronizes from the value $g_2=1.4$. In the figures (a), (b), (c), (d), we represent the phase portrait (u_1,u_2). It is observed (figure (d)) that for $g_2=1.4$, $u_1\approx u_2$, it means that the synchronization occurs.

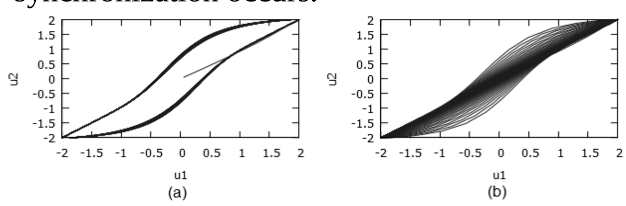

Figure 2. Synchronization of a complete network of two nonlinearly coupled neurons in the phase portrait (u_1,u_2). The synchronization occurs for $g_2=1.4$. Before synchronization, for $g_2=0.0001$, the figure (a) represents the temporal dynamic of u_2 with respect to u_1; the figure (b) represents the temporal dynamic of u_2 with respect to u_1 for $g_2=0.01$; the figure (c) represents the temporal dynamic of u_2 with respect to u_1 for $g_2=0.5$. Figure (d), for $g_2=1.4$, the synchronization occurs since $u_1\approx u_2$

From the above result, in the case of two nonlinearly coupled neurons, for the coupling strength over or equal to $g_2 = 1.4$ these neurons have a synchronous behavior (Figure 2d). By doing similarly for the complete networks of nonlinearly identical coupled neurons, the values of coupling strength according to the number of neurons n are reported in Table 1.

Table 1. The minimal coupling strength necessary to observe the synchronization of n nonlinearly coupled neurons

n		2	3	4	5
g_n		1.4	0.933	0.7	0.56
n	6	7	8	9	10
g_n	0.467	0.4	0.35	0.311	0.28
n	11	12	13	14	15
g_n	0.255	0.233	0.215	0.2	0.187
n	16	17	18	19	20
g_n	0.175	0.165	0.156	0.147	0.14
n	21	22	23	24	25
g_n	0.133	0.127	0.122	0.117	0.112
n	26	27	28	29	30
g_n	0.108	0.104	0.1	0.097	0.093
n	31	32	33	34	35
g_n	0.09	0.088	0.085	0.082	0.08
n	36	37	38	39	40
g_n	0.079	0.076	0.074	0.072	0.07

Following these numerical experiments, it is easy to see that the coupling strength required for observing the synchronization of n neurons depends on the number of neurons. Indeed, the blue points in Figure 3 represent the coupling strength of synchronization according to the number of neurons in complete network from Table 1, and we can find a function depending on the number of neurons represented by the red curve given by the following equation:

$$g_n = \frac{2g_2}{n-1}, \qquad (6)$$

where n is the number of neurons in the network and g_2 is the coupling strength necessary to get the synchronization of 2 coupled complete network. Therefore, the coupling strength necessary to obtain the synchronization in the complete network decreases while the number of neurons increases following the law (6).

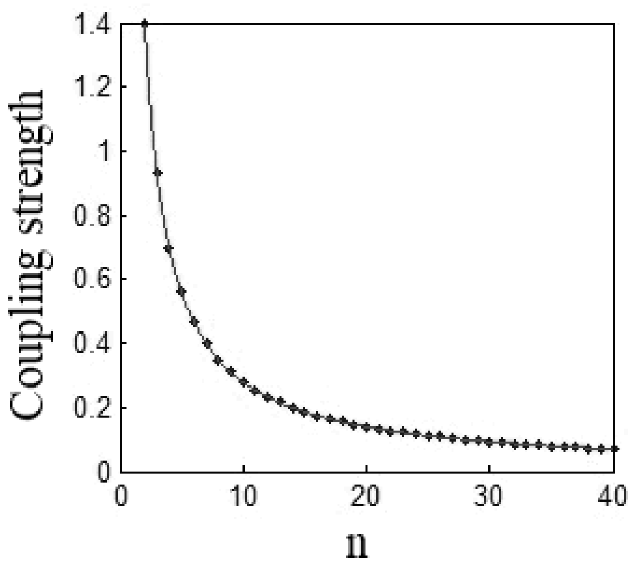

Figure 3. The evolution of the coupling strength for which the synchronization of neurons takes place according to the number nonlinearly coupled neurons in complete network and it follows the law $g_n = \frac{2g_2}{n-1}$

4. Conclusion

This study gave the sufficient condition on the coupling strength to achieve the synchronization in a complete network of n coupled dynamical systems of Fitzhugh-Nagumo type. Theorem 1 shows that the bigger the value of n is, the smaller the g_n is. Numerically, it displays that the synchronization is stable when the coupling strength exceeded to certain threshold and depends on the number of "neurons" in graphs. The bigger the number of "neurons" is, the easier the phenomenon of synchronization will be obtained. Then, a compromise between the theoretical and numerical results can be reached. In addition, it is necessary to conduct further studies on the different synchronization regimes in free networks coupled by chemical synapse.

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