THE DECOMPOSITION OF CYCLIC MODULES IN WEIGHTED LEAVITT PATH ALGEBRA OF REDUCIBLE GRAPH

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Abstract

In this paper, we describe the structure of the cyclic module in the weighted Leavitt path algebra of reducible weighted graph generated by the elements in the induced graph.

Keywords: *Cyclic module, simple module, weighted Leavitt path algebra.*

PHÂN TÍCH MÔĐUN CYCLIC TRONG ĐẠI SỐ ĐƯỜNG ĐI LEAVITT CÓ TRỌNG SỐ CỦA CÁC ĐỒ THỊ KHẢ QUY

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Tóm tắt

Trong bài viết này, chúng tôi mô tả cấu trúc của môđun cyclic trong đại số đường đi Leavitt có trọng số của các đồ thị khả quy sinh bởi các phần tử trong đồ thị cảm sinh.

Từ khóa: *Môđun cyclic, môđun đơn, đại số đường đi Leavitt có trọng số.*

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1. Introduction

Let E be a (directed) graph and K a field, Abrams and Aranda Pino (2005) introduced the *Leavitt path algebra* $L_K(E)$ induced by E. These Leavitt path algebras is a generalization of the Leavitt algebras $L_K(1,n)$ (Leavitt, 1962). Hazrat and Preusser (2017) introduced the generalization of algebra $L_K(E)$ constructed by weighted graph (E,w), called *weighted Leavitt path algebra,* denoted by $L_K(E,w)$. This weighted Leavitt path algebras is a generalisation of the algebras $L_K(m,n)$ (Leavitt, 1962) and if (E,w) is an unweighted graph, i.e, weighted map $w=1$ then $L_K(E,w)$ is the usual $L_K(E)$.

In module theory, the structure of a module or the structure of a submodule is commonly described. According to this study, there are two important classes of modules, simple and cyclic module. Simple module is a non-zero module and has no non-zero proper submodule. Cyclic module is a module generated by one element. By describing the structure of the cyclic module generated by the edge and vertex of the original induced graph, (Ngo et al., 2020) showed that in the case of Leavitt path algebra, the cyclic module is generally not a simple module. In this paper, we describe the structure of the cyclic module in the weighted Leavitt path algebra generated by the edges and vertices of the reducible weighted graph.

2. Leavitt path algebras

In this section, we recall some concepts of directed graphs and Leavitt path algebras.

A (directed) graph $E=(E^0,E^1,s,r)$ consists of two disjoint sets E^0 and E^1, called *vertices* and *edges* respectively, together with two maps $r,s:E^1\to E^0$. The vertices $s(e)$ and $r(e)$ serve as the *source* and the *range* of the edge e, respectively. A graph E is *finite* if both sets E^0 and E^1 are finite. In this paper, all of the graphs are finite.

A *path* $p=e_1e_2...e_n$ in a graph E is a sequence of edges $e_1,e_2,...,e_n$ such that $r(e_i)=s(e_{i+1})$ for $i=1,2,...,n-1$. In this case, we say that the path p starts at the vertex $s(p):=s(e_1)$ and ends at the vertex $r(p):=r(e_n)$.

For each $v\in E^0$, we denote the set

$T(v)=\{u\in E^0:\exists p,\ s(p)=v,\ r(p)=u\}$ is the tree of v.

Vertex v is called a *sink* if $s^{-1}(v)=\varnothing$; vertex v is called a *source* if $r^{-1}(v)=\varnothing$, vertex v is *isolated* if it is both a source and a sink; and vertex v is *regular* if $0<s^{-1}(v)<\infty$.

A *weighted graph* $(E,w)=(E^0,E^{st},s,r,w)$ consists of three countable sets, E^0 called *vertices,* E^{st} *structured edges* and E^1 *edges,* maps $r,s:E^{st}\to E^0$, and a *weight map* $w:E^{st}\to\mathbb{N}^*$ such that

$$E^1=\bigcup_{\alpha\in E^{st}}\{\alpha_i\,|\,1\le i\le w(\alpha)\},$$

i.e., for any $\alpha\in E^{st}$, with $w(\alpha)=k$, there are k distinct elements $\{\alpha_1,...,\alpha_k\}$, and E^1 is the disjoint union of all such sets for all $\alpha\in E^{st}$.

If $w:E^{st}\to\mathbb{N}^*$ is the constant map $w(\alpha)=1$ for all $\alpha\in E^{st}$, then (E,w) is the usual unweighted graph. For each $v\in E^0$, we denote

$$w(v)=\max\{w(\alpha)_{\alpha\in s^{-1}(v)}\}$$

called that *weight of v.*

Let $(E,w)=(E^0,E^{st},s,r,w)$ be a weighted graph, we denote $\overline{E}=(\overline{E}^0,\overline{E}^1,\overline{r},\overline{s})$ by the unweighted graph associated with (E,w), where

$$\overline{E}^0=E^0,\ \overline{E}^1=\bigcup_{\alpha\in E^{st}}\{\alpha_1,...,\alpha_{w(\alpha)}\},$$

$$\overline{r}(\alpha_i)=r(\alpha),\ \overline{s}(\alpha_i)=s(\alpha),1\le \mathrm{i}\le \mathrm{w}(\alpha),\ \alpha\in \mathrm{E}^{st}.$$

For an arbitrary graph $E=(E^0,E^1,s,r)$ and an arbitrary field K, the *Leavitt path algebra* $L_K(E)$ of the graph E *with coefficients* *in K* is the *K*-algebra generated by the union of the set E^0 and two disjoint copies of E^1, say E^1 and $\{e^* \mid e \in E^1\}$, satisfying the following relations for all $v, w \in E^0$ and $e, f \in E^1$:

(1) $vw = \delta_{v,w} w$, (where δ is the Kronecker delta);

(2) $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$;

(3) $e^* f = \delta_{e,f} r(e)$;

(4) $v = \sum_{e \in s^{-1}(v)} ee^*$ for any regular vertex v.

For an arbitrary weighted graph $(E, w) = (E^0, E^{st}, s, r, w)$ and an arbitrary field *K*, the *weighted Leavitt path algebra* $L_K(E, w)$ of the graph (E, w) *with coefficients in K* is the *K*-algebra generated by the union of the set $\overline{E}^0$ and two disjoint copies of $\overline{E}^1$ say $\overline{E}^1$ and $\{e^* \mid e \in \overline{E}^1\}$, satisfying the following relations for all $v, w \in \overline{E}^0$, $\alpha, \beta \in E^{st}$ and $e \in \overline{E}^1$:

(1') $vw = \delta_{v,w} w$;

(2') $s(e)e = e = er(e)$ and $r(e)e^* = e^* = e^* s(e)$;

(3') $\sum_{\alpha \in s^{-1}(v)} \alpha_i \alpha_j^* = \delta_{i,j} v$;

(4') $\sum_{1 \le i \le w(v)} \alpha_i^* \beta_i = \delta_{\alpha,\beta} r(\alpha)$, $\alpha, \beta \in s^{-1}(v)$ for any regular vertex v.

3. Main results

In this section, we describe the structure of the cyclic module in the weighted Leavitt path algebra of the reducible weighted graph generated by the edges and vertices of the original graph.

Let M be a left module over the ring R (in this paper, all modules are left modules). M is called *cyclic module* if M is generated by only one element. M is called a *simple module* if it is the non-zero module and has no non-zero proper submodules. The following results were presented by Ngo et al. (2020):

Theorem 1. *Let* $E = (E^0; E^1, r, s)$ *be a finite graph and* K *an arbitrary field. Then, for each* $f \in E^1$ *where* $\left|s^{-1}(r(f))\right| > 0$, *we have*

$$fL_K(E) = \bigoplus_{e \in s^{-1}(r(f))} feL_K(E).$$

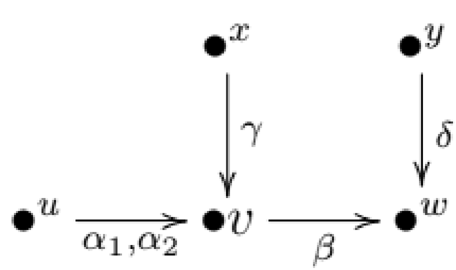
Theorem 2. *Let* $E = (E^0; E^1, r, s)$ *be a finite graph and* K *an arbitrary field. Then, for each* $v \in E^0$ *which is not a sink, we have*

$$vL_K(E) \cong \bigoplus_{e \in s^{-1}(v)} r(e) L_K(E).$$

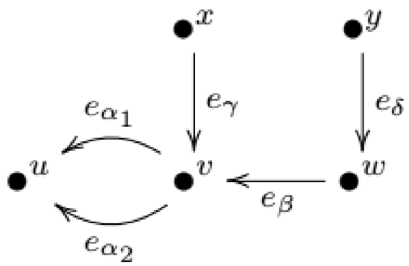
An element $\alpha \in E^{st}$ is called *weighted* if $w(\alpha) > 1$ and *unweighted* otherwise. The set of all weighted elements of E^{st} is denoted by E_w^{st}. An element $v \in E^0$ is called *weighted* if $w(v) > 1$ and *unweighted* otherwise. The set of all weighted elements of E^0 is denoted by E_w^0. The set $\overline{E_w^0} = \bigcup_{v \in E_w^0} T(v)$ is called *weight forest of* (E, w). A weighted graph (E, w) with $E_w^0 \neq \emptyset$ is called *reducible* if $|s^{-1}(v)| \le 1$ and $|r^{-1}(v) \cap s^{-1}(E_w^0)| \le 1$ for any $v \in E_w^0$ and *irreducible* otherwise.

Let (E, w) be a weighted graph. We construct an unweighted graph $F = (F^0, F^1, s', r')$ as follows. Let $F^0 = E^0$, $F^1 = \{e_{\alpha_i} : \alpha_i \in \overline{E}^1\}$, $s'(e_{\alpha_i}) = s(\alpha)$ and $r'(e_{\alpha_i}) = r(\alpha)$ if $s(\alpha) \notin \overline{E_w^0}$, $s'(e_{\alpha_i}) = r(\alpha)$ and $r'(e_{\alpha_i}) = s(\alpha)$ if $s(\alpha) \in \overline{E_w^0}$. The graph F is called *the unweighted graph associated with* (E, w).

Example 1. The following weighted graph is reducible:



and the unweighted graph associated with it is



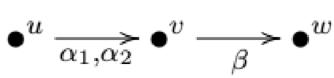
Lemma 1. *Let* (E,w) *be a reducible weighted graph. Consider the weighted graph* (E',w') *one gets by dropping all vertices which do not belong to the weight forest* $\overline{E_w^0}$, *and all structured edges* α *such that* $s(\alpha)$ *or* $r(\alpha)$ *does not belong to the weight forest* $\overline{E_w^0}$. *Then, all of the connected components of* (E',w') *are either circle graphs or oriented line graphs.*

Proof. For each vertex $v\in\overline{E_w^0}$, we can check easily that

$$|s^{-1}(v)|\le 1 \text{ and } |r^{-1}(v)\cap s^{-1}(E_w^0)|\le 1.$$

This implies that for each vertex v in (E',w'), we get $|s^{-1}(v)|\le 1$ and $|r^{-1}(v)|\le 1$. This concludes the proof. □

Example 2. Consider the weighted graph (E,w) as in Example 1, we get (E',w') is oriented line



The following result is from Hazrat and Preusser (2017):

Lemma 2. *If* (E,w) *is a reducible weighted graph and* F *is the unweighted graph associated with* (E,w). *Then,* $L_K(E,w)\cong L_K(F)$.

We are finally in a position to establish the main result of this article.

Theorem 3. *Let* (E,w) *is a reducible weighted graph,* F *is the unweighted graph associated with* (E,w) *and* K *an arbitrary field. Then,*

(a) For each edge $f\in E^{st}$, *we have:*

$$fL_K(E,w)=\begin{cases}\bigoplus\limits_{1\le i\le w(\alpha)} e_f e_{\alpha_i} L_K(F), \alpha\in r^{-1}(s(f)),\ f\in(E',w')\\ \bigoplus\limits_{\beta\in s^{-1}(r(f))} e_f e_\beta L_K(F),\ f\notin(E',w')\end{cases}$$

(b) For each vertex $v\in E^0$, *we have:*

$$vL_K(E,w)=\begin{cases}\bigoplus\limits_{1\le i\le w(\alpha)} r(e_{\alpha_i})L_K(F), \alpha\in r^{-1}(v),\ v\in\overline{E_w^0}\\ \bigoplus\limits_{\beta\in s^{-1}(v)} r(e_\beta)L_K(F),\ v\notin\overline{E_w^0}\end{cases}$$

Proof. It follows from Lemma 1 that, if $f\notin(E',w')$ then for any edge $\beta\in s^{-1}(r(f))$, the weight of β be 1; if $v\notin\overline{E_w^0}$ then for any edge $\beta\in s^{-1}(v)$ the weight of β be 1.

Consider the map

$$\varphi:\overline{E}^0\cup\overline{E}^1\cup(\overline{E}^1)^*\to F^0\cup F^1\cup(F^1)^*$$

where

$$\varphi(v)=v,\ \forall v\in\overline{E}^0;$$

$$\varphi(\alpha_i)=e_{\alpha_i},\ \varphi(\alpha_i^*)=e_{\alpha_i}^*,\ \forall\alpha_i\in\overline{E}^1 \text{ such that } s(\alpha)\notin\overline{E_w^0}$$

and

$$\varphi(\alpha_i)=e_{\alpha_i}^*,\ \varphi(\alpha_i^*)=e_{\alpha_i},\ \forall\alpha_i\in\overline{E}^1 \text{ such that } s(\alpha)\in\overline{E_w^0}.$$

Then, φ can be continued to the isomorphism $L_K(E,w)\cong L_K(F)$ as in Lemma 2. In the usual Leavitt path algebra $L_K(F)$, we apply Theorem 1 and Theorem 2 to get the result. □

Example 3. Consider the reducible weighted graph (E,w) with the associated graph F as in Example 1. Then, with an arbitrary field K, we have

a) $\delta L_K(E,w)\cong e_\delta e_\beta L_K(F)$ and

$$\beta L_K(E,w)\cong e_\beta e_{\alpha_1}L_K(F)\oplus e_\beta e_{\alpha_2}L_K(F);$$

b) $yL_K(E,w)\cong wL_K(F)\cong vL_K(F)$

$$\cong vL_K(E,w)\cong uL_K(F)\oplus uL_K(F)$$

4. Conclusion

In this paper, we show a criterion for a weighted graph which is reducible. Consequently, we describe the structure of a cyclic module in the weighted Leavitt path algebra generated by the edges and vertices of the reducible weighted graph. These results also show that in the case of weighted Leavitt path algebra, the cyclic module is not a simple module in general.

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