ROBUST STABILITY OF SWITCHED POSITIVE LINEAR SYSTEMS

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Article history

Received: 20/4/2021; Received in revised form: 24/5/2021; Accepted: 12/7/2021

Abstract

The aim of the present paper is to give robust stability based on exponentially stable switched positive linear systems. Our theoretical analysis shows that if there exists a positive stable system of all positive subsystems, then a lower bound and upper bound for stability radius of the switched system under positive affine perturbations are established. In the particular case of two dimensional switched system, including two switching signals, we obtain a formula of stability radius. Several examples are provided to illustrate our approach.

Keywords: *Positive linear systems, robust stability, stability radii, switched linear systems.*

TÍNH ỔN ĐỊNH VỮNG CỦA HỆ CHUYỂN MẠCH TUYẾN TÍNH DƯƠNG

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Lịch sử bài báo

Ngày nhận: 20/4/2021; Ngày nhận chỉnh sửa: 24/5/2021; Ngày duyệt đăng: 12/7/2021

Tóm tắt

Mục tiêu của bài báo này nghiên cứu tính ổn định vững dựa trên ổn định mũ của hệ chuyển mạch tuyến tính dương. Phân tích lý thuyết của chúng tôi chỉ ra rằng nếu tồn tại một hệ tuyến tính dương ổn định là một chặn của tất cả các hệ dương con, thì cận dưới và cận trên cho bán kính ổn định này đối với các nhiễu cấu trúc affine dương của hệ thống được thiết lập. Trong trường hợp đặc biệt đối với hệ thống chuyển mạch hai chiều có hai tín hiệu chuyển mạch, chúng tôi thu được công thức bán kính ổn định. Một số ví dụ được cung cấp để minh họa cách tiếp cận của chúng tôi.

Từ khóa: Hệ tuyến tính dương, ổn định vững, bán kính ổn định, hệ chuyển mạch tuyến tính

DOI: https://doi.org/10.52714/dthu.10.5.2021.891

Cite: Le, V. N. (2021). Robust stability of switched positive linear systems. *Dong Thap University Journal of Science*, *10*(5), 15-24. https://doi.org/10.52714/dthu.10.5.2021.891.

1. Introduction

A *switched system* is described by a family of subsystems and a rule that controls the switching between them. Switched systems have gained attention from many scientists since they can be applied in a wide variety of tasks, including mechanical engineering, the automotive industry, power systems, aircraft traffic, and many other fields. The books (Liberzon, 2003; Sun & Ge, 2011) contain reports on various theoretical developments for switched systems as well as their applications in some of these areas. In the mathematical setting, such a system, in the case of a linear continuous-time model, can be described by a linear time-varying differential equation of the form

$$\dot{x}(t) = A_{\sigma(t)} x(t), \ t \ge 0, \ \sigma \in \Sigma \qquad (1)$$

Where $A_{\sigma(t)} \in \mathcal{A} := \{A_k \in K^{n\times n}, k \in \underline{N}\}, t \ge 0$, $\underline{N} := \{1, 2, ..., N\}$ a given family of N matrices with elements in $K, K = \mathbb{C}$ or $\mathbb{R}$ and Σ is a set of switching signals $\sigma : [0, +\infty) \to \underline{N}$, which are piecewise constant right-side continuous functions with points of discontinuity t_i, $i = 1, 2, ...$ satisfying $\tau := \inf_{i\in\mathbb{N}} (t_{i+1} - t_i) > 0$.

Among qualitative properties of switched systems, stability and stabilization play a pivotal role and have been most widely investigated. To mention a few, we refer the reader to monographs (Liberzon, 2003; Sun & Ge, 2011), survey papers (Lin & Antsaklis, 2009; Shorten et al., 2007), and the references therein. One of the basic problems in stability analysis of switched systems is to find conditions guaranteeing stability/stabilizability *under arbitrary switching*. It has been well established, for instance, that the zero solution of the switched linear system (1) is exponentially stable under arbitrary switching signal $\sigma \in \Sigma$ if all subsystems

$$\dot{x}(t) = A_k x(t), \ t \ge 0, \ k \in \underline{N} \qquad (2)$$

have a common quadratic Lyapunov function (or QLF, for short) of the form $V(x) = x^T P x$ or (see Blanchini et al., 2015; Ding et al., 2011) a common co-positive Lyapunov linear function of the form $V(x) = v^T x$.

The estimations of stability radii of switched linear systems and periodically switched linear systems were introduced in the paper (Nguyen et al., 2020; Do et al., 2019). The stability radii of the positive linear system proposed by Son-Hinrichsen (see Nguyen et al., 1996) has a real stability radius equal to the complex stability radius, while the switched positive linear system interested by many authors and given conditions stable (see Blanchini et al., 2015; Ding et al., 2011; Gurvits et al., 2007; Mason et al., 2007; Le et al., 2020; Sun., 2016), have studied robust stability in (Le et al., 2020). However, with an inevitable limitation, the formula of stability radius has not been fully studied. In this paper, we seek to provide a case with a stability radius.

2. Preliminaries

Let denote $\mathbb{R}^{n\times m}$, $\mathbb{R}_+^{n\times m}$ be a set of all n×m matrices with elements in $\mathbb{R}, \mathbb{R}_+$ respectively. We adopt the notation $A \ge 0$ for the case that a matrix A with entries is *non-negative*. A non-negative matrix with at least one positive entry is a positive matrix, denoted as (A>0). On the other hand, if all entries of matrix A are positive, then A is *strictly positive* ($A \gg 0$). Given two matrices A and B, of the same size $A \ge B, A > B$ and $A \gg B$ are synonymous of $A - B \ge 0, A - B > 0$ and $A - B \gg 0$, respectively. Throughout this article, unless otherwise stated, the norm of a matrix $A \in K^{n\times m}$ is understood as its operator norm induced by a given pair of monotonic vector norms on K^n and K^m that is $\| A \| = \max\{\| Ax \| : \| x \| = 1\}$.

For any matrix $A \in \mathbb{R}^{n\times n}$, the spectral abscissa of A is denoted by

$\mu(\mathrm{A}) = \max\{\mathrm{Re}\lambda : \lambda \in \sigma(\mathrm{A})\}$, where $\sigma(\mathrm{A}) := \{z \in \mathbb{C} : \det(z\mathrm{I}_n\text{-}\mathrm{A}) = 0\}$ is the set of all eigenvalues of A. A matrix $\mathrm{A} \in \mathbb{R}^{n\times n}$ is called Hurwitz if and only if $\mu(\mathrm{A})<0$. A matrix $\mathrm{A} \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if all off-diagonal elements of A are non-negative. Consider a linear continuous-time system in K^n the form:

$$\dot{x}(t) = \mathrm{A}x(t);\ x \in \mathrm{K}^n;\ t \ge 0. \quad (3)$$

Assume that the system (3) is Hurwitz stable that is $\mu(\mathrm{A}) < 0$ and is subjected to structured affine perturbations of the form:

$$\dot{x}(t) = (\mathrm{A}+\mathrm{D}\Delta\mathrm{E})x(t);\ x \in \mathrm{K}^n;\ t \ge 0, \quad (4)$$

where $\Delta \in \mathrm{K}^{l\times q}$ is unknown disturbance matrix, $\mathrm{D} \in \mathrm{K}^{n\times 1}, \mathrm{E} \in \mathrm{K}^{q\times n}$ are given matrices defining the structured perturbations. We have the well-known notion of structured stability radius (Hinrichsen and Pritchard., 1986) which is defined, for $\mathrm{K}= \mathbb{R};\mathbb{C}$, as:

$$r_{\mathrm{K}}(\mathrm{A};\mathrm{D};\mathrm{E}) := \inf\{\|\Delta\| : \Delta \in \mathrm{K}^{l\times q},\ \mathrm{A} + \mathrm{D}\Delta\mathrm{E} \text{ not Hurwitz stable}\}. \quad (5)$$

In particular, as the equation is shown in (Nguyen Khoa Son and Hinrichsen, 1996), if $\mathrm{A} \in \mathbb{R}^{n\times n}$ is a Metzler Hurwitz stable and $D \in \mathbb{R}_+^{n\times l}; E \in \mathbb{R}_+^{q\times n}$, then formula (5) is computed as follow:

$$r_{\mathbb{C}}(\mathrm{A,D,E}) = r_{\mathbb{R}}(A, D, E) = \frac{1}{\|\mathrm{EA}^{-1}\mathrm{D}\|}. \quad (6)$$

We repeat the following two theorems about Metzler matrices.

Theorem 2.1. (Nguyen et al., 1996, Proposition 1). Suppose that $\mathrm{A} \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then

(i) (Perron-Frobenius) $\mu(\mathrm{A})$ is an eigenvalue of A, and there exists a non-negative eigenvector $x \neq 0$ such that $\mathrm{A}x = \mu(\mathrm{A})x$.

(ii) Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \ge 0$ such that $\mathrm{A}x \ge \alpha x$ if and only if $\mu(\mathrm{A}) \ge \alpha$.

(iii) $(t\mathrm{I}_n\text{-}\mathrm{A})^{-1}$ exists and is non-negative if and only if $t > \mu(\mathrm{A})$.

The following result is immediate from Theorem 2.1.

Theorem 2.2. Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent:

(i) $\mu(\mathrm{A}) < 0$;

(ii) $\mathrm{A}p \ll 0$ for some $p \in \mathbb{R}_+^n, p \gg 0$;

(iii) A is invertible and $\mathrm{A}^{-1} \le 0$.

The following lemma is reused in that section.

Lemma 2.1 (Do Duc Thuan *et al.*, 2019, Lemma 2.4). Let α, β, γ be given positive numbers, and

$$\Omega := \{(x, y) \in \mathbb{R}^2 : 2xy + \alpha x + \beta y - \gamma \ge 0, x \ge 0,\ y \ge 0\}.$$

Then,

$$\min_{(x,y)\in\Omega}\{x + y\} = \begin{cases} \sqrt{\alpha\beta + 2\gamma} - \dfrac{\alpha+\beta}{2} & \text{if } \alpha\beta + 2\gamma > \omega^2, \\ \dfrac{\gamma}{\beta} & \text{if } \alpha\beta + 2\gamma \le \omega^2 \text{ and } \omega = \beta > \alpha, \\ \dfrac{\gamma}{\alpha} & \text{if } \alpha\beta + 2\gamma \le \omega^2 \text{ and } \omega = \alpha > \beta, \end{cases}$$

where $\omega = \max\{\alpha, \beta\}$.

The following result is immediate from Lemma 2.2.

Lemma 2.2. Let α, β be given nonnegative numbers, $\gamma > 0$ and $\Omega := \{(x, y) \in \mathbb{R}^2 : 2xy + \alpha x + \beta y - \gamma \ge 0, x \ge 0,\ y \ge 0\}$, set $\omega = \max\{\alpha, \beta\}$. Then,

$$\min_{(x,y)\in\Omega}\{x+y\}=\begin{cases}\sqrt{\alpha\beta+2\gamma}-\dfrac{\alpha+\beta}{2} \text{ if } \alpha\beta+2\gamma>\omega^2;\alpha,\beta>0,\\ \dfrac{\gamma}{\beta} \text{ if } \alpha\beta+2\gamma\le\omega^2 \text{ and } \omega=\beta>\alpha>0,\\ \dfrac{\gamma}{\alpha} \text{ if } \alpha\beta+2\gamma\le\omega^2 \text{ and } \omega=\alpha>\beta>0,\\ 2.\sqrt{\dfrac{\gamma}{2}} \text{ if } \alpha=\beta=0,\\ \sqrt{2\gamma}-\dfrac{\alpha}{2} \text{ if } \beta=0 \text{ and } 2\gamma-\alpha^2>0,\\ \sqrt{2\gamma}-\dfrac{\beta}{2} \text{ if } \alpha=0 \text{ and } 2\gamma-\beta^2>0.\end{cases}$$

Consider a continuous-time switched positive linear system in $\mathbb{R}^n$ described by (1). This ensures that if $x(0)=x_0$ belongs to the positive orthant $\mathbb{R}^n_+$, then for any switching signal $\sigma\in\Sigma$ the system (1) admits a unique solution $x(t,x_0,\sigma)$, $t\ge 0$.

Definition 2.1 (Ding et al., 2011, Definition 2.4). The switched linear system (1) is said to be positive if $x_0\ge 0$ implies that $x(t,x_0,\sigma)\ge 0$ for all $t\ge 0$.

Definition 2.2 (Blanchini et al., 2015, Definition 3.1). The switched positive linear system (1) is said to be exponentially stable if there exist real constants $M>0$ and $\beta>0$ such that all the solutions of (1) satisfy

$$\|x(t,x_0,\sigma)\|\le Me^{-\beta t}\|x_0\|, \quad (7)$$

for every $x_0\in\mathbb{R}^n_+, t\ge 0$ and for all switching signal $\sigma\in\Sigma$.

Lemma 2.3 (Ding et al., 2011, Lemma 2.3). The switched positive linear system (1) is positive if and only if A_k, $k\in\underline{N}$ are Metzler matrices.

Lemma 2.4 (Blanchini et al., 2015, Definition 3.1). Consider a switched positive linear system described by (1). If there exists $v\in\mathbb{R}^n_+, v\gg 0$ such that:

$$v^T A_k \ll 0, \ \forall k\in\underline{N}, \quad (8)$$

then the positive switched linear system (1) is exponentially stable.

Lemma 2.5 (Nguyen Khoa Son et al., 2020, Lemma 2). Consider a switched positive linear system described by (1). If there exists $v\in\mathbb{R}^n_+, v\gg 0$ such that:

$$A_k v\ll 0, \ \forall k\in\underline{N}, \quad (9)$$

then the switched positive linear system (1) is exponentially stable. Given Theorem 2.2 (ii), the preceding result immediately implies.

Corollary 2.1. If there exists a Hurwitz stable Metzler matrix A_0 such that:

$$A_k\le A_0, \forall k\in\underline{N}, \quad (10)$$

then the switched positive linear system (1) is exponentially stable for any $\sigma\in\Sigma$.

Theorem 2.3. (Ding *et al.*, 2011, Theorem 4.2). Let $A_1=(a^1_{ij}), A_2=(a^2_{ij})\in\mathbb{R}^{2\times 2}$, $i,j=1,2$ be Metzler and Hurwitz matrices. The following statements are equivalent.

(i) The switched positive linear system (1) is exponentially stable.

(ii) The switched positive linear subsystems (2) have a common linear copositive Lyapunov function (CLCLF).

(iii) There exists a diagonal matrix P > 0 such that $A_k^T P+PA_k<0$ for k = 1, 2.

(iv) $\begin{vmatrix} a^1_{11} & a^2_{21}\\ a^1_{12} & a^2_{22}\end{vmatrix}>0, \begin{vmatrix} a^2_{11} & a^1_{21}\\ a^2_{12} & a^1_{22}\end{vmatrix}>0.$

We prove the following lemma.

Lemma 2.6. Let $A\in\mathbb{R}^{2\times 2}$ be Metzler Hurwitz stable. Then, $a_{ii}<0, i=1,2.$

Proof. Consider is the determinant

$$\det(A-\lambda I) =\begin{vmatrix} a_{11}-\lambda & a_{21}\\ a_{12} & a_{22}-\lambda\end{vmatrix} =\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21}).$$

Assume $\det(A-\lambda I)=0$, we have

$$\lambda_{1,2}(A)=\frac{a_{11}+a_{22}}{2}\pm\frac{\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2}$$

Since A is Metzler Hurwitz stable, then $\lambda_{1,2}<0$ and

$(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})=(a_{11}-a_{22})^2+4a_{12}a_{21}\geq 0.$
We obtain

$$\begin{cases}\lambda_1+\lambda_2=a_{11}+a_{22}<0\\ \lambda_1\lambda_2=a_{11}a_{22}-a_{12}a_{21}>0\\ a_{11}+a_{22}+\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}<0\\ a_{11}+a_{22}-\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}<0 \text{ (Obviously).}\end{cases}$$

Equivalently, $\begin{cases}a_{11}+a_{22}<0\\ a_{11}a_{22}>a_{12}a_{21}\end{cases}.$

Since A is Metzler Hurvitz stable, then $a_{12}\geq 0, a_{21}\geq 0,\ a_{11}a_{22}-a_{12}a_{21}>0.$

We have $\begin{cases}a_{11}+a_{22}<0\\ a_{11}a_{22}>0\end{cases}.$

This implies $a_{11}<0,\ a_{22}<0.$ The proof is completed.

3. Stability radius of switched positive linear systems

In this section, consider a continuous-time switched positive linear system in $\mathbb{R}^n$ described by (1). We assume that the switched positive linear system (1) is exponentially stable and the matrices $\mathrm{A}_k,\ k\in\underline{N}$ of positive subsystems (2) are subjected to affine positive perturbations of the form:

$$\mathrm{A}_k\to\tilde{\mathrm{A}}_k:=\mathrm{A}_k+\mathrm{D}_k\Delta_k\mathrm{E}_k,\ k\in\underline{N}, \quad (11)$$

where $\mathcal{D}:=\{\mathrm{D}_k,\ \mathrm{D}_k\in\mathbb{R}_+^{n\times l_k}\}$, $\mathcal{E}:=\{\mathrm{E}_k,\ \mathrm{E}_k\in\mathbb{R}_+^{q_k\times n}\},\quad k\in\underline{N}$ are given matrices defining the structure of the perturbations and $\Delta_k\in\mathbb{R}_+^{l_k\times q_k},\ k\in\underline{N}$ are unknown disturbances. Then the perturbed system is described by

$$\dot{x}(t)=\tilde{\mathrm{A}}_{\sigma(t)}\,x(t),\ t\geq 0, \sigma\in\Sigma, \qquad (12)$$

$$\tilde{\mathrm{A}}_{\sigma(t)}\in\left\{\mathrm{A}_k+\mathrm{D}_k\Delta_k\mathrm{E}_k,\Delta_k\in\mathbb{R}_+^{l_k\times q_k},\mathrm{k}\in\underline{N}\right\},\ t\geq 0.$$

An important question arising in the robustness analysis of stability for the nominal system (1) subjected to parameter perturbations is how large perturbations $\Delta_k, k\in\underline{N}$ are allowable in the perturbed perturbation (12), without destroying the exponential stability of the system. To deal with this question, let us measure the size of perturbations $\Delta:=(\Delta_1,\Delta_2,...,\Delta_N)\in\mathbb{R}_+^{l_1\times q_1}\times\mathbb{R}_+^{l_2\times q_2}\times\ldots\times\mathbb{R}_+^{l_N\times q_N}$ by

$$\|\Delta\|_S:=\sum_{k=1}^{N}\|\Delta_k\|. \qquad (13)$$

Definition 3.1. If the switched positive linear system (1) is exponentially stable and is subjected to affine perturbations of the form (12). Then its structured stability radius is defined as:

$$r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E}):=\inf\{\|\Delta\|_S:\Delta=(\Delta_1,\Delta_2,...,\Delta_N),\Delta_k\in\mathbb{R}_+^{l_k\times q_k},k\in\underline{N},$$
$$\exists\sigma\in\Sigma \text{ such that (12) is not exponentially stable}\} \quad (14)$$

where the norm $\|.\|_S$ of perturbations Δ is defined by (13). If $\mathrm{D}_k=\mathrm{E}_k=\mathrm{I},\ k\in\underline{N}$ (the case of unstructured perturbations) then we put $r_{\mathbb{R}_+}(\mathcal{A})=r_{\mathbb{R}_+}(\mathcal{A},\mathrm{I},\mathrm{I}).$

Now, we will use Corollary 2.1 to get a more explicit formula for computing a lower bound of the real stability radius $r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E})$ of the switched positive linear systems (1) with a Metzler Hurwitz stable matrix.

Theorem 3.1. Assume that the switched positive linear system (1) is exponentially stable and is subjected to affine perturbations of the form (11). Moreover, if there exists a Hurwitz stable Metzler matrix $\mathrm{A}_0\in\mathbb{R}^{n\times n}$ such that:

$$\mathrm{A}_k\leq\mathrm{A}_0,\ \forall k\in\underline{N}. \qquad (15)$$

Then the stability radius (14) satisfies the following inequality:

$$\frac{1}{\max_{i,j\in\underline{N}}\left\|\mathrm{E}_i\mathrm{A}_0^{-1}\mathrm{D}_j\right\|}\leq r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E})\leq\min_{k\in\underline{N}}\left\{r_{\mathbb{R}}(\mathrm{A}_k,\mathrm{D}_k,\mathrm{E}_k)\right\}, \quad (16)$$

where, for each $k\in\underline{N}$, $r_{\mathbb{R}}\left(\mathrm{A}_k,\mathrm{D}_k,\mathrm{E}_k\right)=\dfrac{1}{\left\|\mathrm{E}_k\mathrm{A}_k^{-1}\mathrm{D}_k\right\|}$ is the stability radius of the positive subsystems (2).

Proof. To prove the upper bound, assume to the contrary that $r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E})>\min_{k\in\underline{N}} r_{\mathbb{R}}(\mathrm{A}_k,\mathrm{D}_k,\mathrm{E}_k).$

It follows that there exists $k_0\in\underline{N}$ such that $r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E})>r_{\mathbb{R}}(\mathrm{A}_{k_0},\mathrm{D}_{k_0},\mathrm{E}_{k_0}).$

By definition (5) of structured stability radius, we can choose a positive perturbation $\Delta=\Delta_{k_0}$ such that $r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E}) > \| \Delta_{k_0} \| > r_{\mathbb{R}}(\mathrm{A}_{k_0},\mathrm{D}_{k_0},\mathrm{E}_{k_0})$ then the subsystem $\dot{x}(t) = \mathrm{A}_{k_0} x(t), t \ge 0$ is not exponentially stable. This implies, however, that the perturbed switched linear system (11) is not exponentially stable under the switching signal $\sigma(t) \equiv \mathrm{k}_0, t \ge 0$ contradicting the definition of stability radius (14).

To prove the lower bound, based on the article by Le et al. (2020), Theorem 3.1, we also have the results in which $\|\Delta\|_S$ is replaced by $\|\Delta\|_{\max}$. The theorem is proved.

Remark 3.1. In case the two dimensions $\mathrm{A}_\mathrm{k} = (a_{ij}^k) \in \mathbb{R}^{2\times 2}, \mathrm{k}=1,2$ are Metzler Hurwitz stable matrices and $\mathrm{D}_\mathrm{k} = (d_{ij}^k) \in \mathbb{R}_+^{2\times l_k}$, $\mathrm{E}_\mathrm{k} = (e_{ij}^k) \in \mathbb{R}_+^{q_k \times 2}$ are given matrices defining the structure of positive perturbations. We measure the size of the positive structured perturbations $\Delta := (\Delta_1,\Delta_2) \in \mathbb{R}_+^{l_1\times q_1} \times \mathbb{R}_+^{l_2\times q_2}$, $\Delta_\mathrm{k} = (\delta_{ij}^k) \in \mathbb{R}_+^{l_k\times q_k}$; $k,i,j=1,2$ identified by

$$\| \Delta \|_S := \| \Delta_1 \| + \| \Delta_2 \| . \qquad (17)$$

Put $r_1 = \min \| \Delta \|_S$ such that $\| \Delta_1 \| \ge 0,\ \| \Delta_2 \| \ge 0$,

$(d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2) \| \Delta_1 \| . \| \Delta_2 \| +$
$+(a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1) \| \Delta_1 \| +$
$+(a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2) \| \Delta_2 \| - (a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2) \ge 0$ and
$r_2 = \min \| \Delta \|_S$ such that $\| \Delta_1 \| \ge 0,\ \| \Delta_2 \| \ge 0$,
$(d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 + d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2) \| \Delta_1 \| . \| \Delta_2 \| +$
$+(a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1) \| \Delta_1 \| +$
$+(a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2) \| \Delta_2 \| - (a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2) \ge 0.$

Then, we obtain the formula of the stable radius of the system (1) by the following theorem.

Theorem 3.2. Assume that the switched positive linear system (1) is exponentially stable under structured perturbation (12). Then the stability radius is

$$r_{\mathbb{R}_+}(\mathcal{A},\mathcal{D},\mathcal{E}) = \min\{r_1, r_2\}. \qquad (18)$$

Proof. Using Theorem 2.3, item (iv), the perturbed system (11) is not exponentially stable if and only if

$$\begin{vmatrix} a_{11}^1 + d_{11}^1\delta_{11}^1 e_{11}^1 & a_{21}^2 + d_{21}^2\delta_{21}^2 e_{21}^2 \\ a_{12}^1 + d_{12}^1\delta_{12}^1 e_{12}^1 & a_{22}^2 + d_{22}^2\delta_{22}^2 e_{22}^2 \end{vmatrix} \le 0$$

$$\text{or } \begin{vmatrix} a_{11}^2 + d_{11}^2\delta_{11}^2 e_{11}^2 & a_{21}^1 + d_{21}^1\delta_{21}^1 e_{21}^1 \\ a_{12}^2 + d_{12}^2\delta_{12}^2 e_{12}^2 & a_{22}^1 + d_{22}^1\delta_{22}^1 e_{22}^1 \end{vmatrix} \le 0.$$

Equivalently

$$\left[\begin{array}{l} (a_{11}^1 + d_{11}^1\delta_{11}^1 e_{11}^1)(a_{22}^2 + d_{22}^2\delta_{22}^2 e_{22}^2) - \\ -(a_{12}^1 + d_{12}^1\delta_{12}^1 e_{12}^1)(a_{21}^2 + d_{21}^2\delta_{21}^2 e_{21}^2) \le 0; \\ (a_{11}^2 + d_{11}^2\delta_{11}^2 e_{11}^2)(a_{22}^1 + d_{22}^1\delta_{22}^1 e_{22}^1) - \\ -(a_{12}^2 + d_{12}^2\delta_{12}^2 e_{12}^2)(a_{21}^1 + d_{21}^1\delta_{21}^1 e_{21}^1) \le 0 \end{array}\right.$$

or equivalently,

$$\left[\begin{array}{l} a_{11}^1 a_{22}^2 + a_{11}^1 d_{22}^2 e_{22}^2 \delta_{22}^2 + a_{22}^2 d_{11}^1 e_{11}^1 \delta_{11}^1 + \\ +d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 \delta_{11}^1 \delta_{22}^2 - (a_{12}^1 a_{21}^2 + a_{12}^1 d_{21}^2 e_{21}^2 \delta_{21}^2) \\ -(a_{21}^2 d_{12}^1 e_{12}^1 \delta_{12}^1 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 \delta_{12}^1 \delta_{21}^2) \le 0; \\ a_{11}^2 a_{22}^1 + a_{11}^2 d_{22}^1 e_{22}^1 \delta_{22}^1 + a_{22}^1 d_{11}^2 e_{11}^2 \delta_{11}^2 + \\ +d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 \delta_{11}^2 \delta_{22}^1 - (a_{12}^2 a_{21}^1 + a_{21}^1 d_{12}^2 e_{12}^2 \delta_{12}^2) \\ -(a_{12}^2 d_{21}^1 e_{21}^1 \delta_{21}^1 + d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 \delta_{21}^1 \delta_{12}^2) \le 0. \end{array}\right.$$

Since $\mathrm{D}_\mathrm{k} \in \mathbb{R}_+^{n\times l_k}$, $\mathrm{E}_\mathrm{k} \in \mathbb{R}_+^{q_k\times n}; \Delta_\mathrm{k} \in \mathbb{R}_+^{l_k\times q_k}$ and A_k, k=1,2 are Metzler Hurwitz matrices the $a_{ii}^k < 0$ and $a_{ij}^k \ge 0, \forall i \ne j = 1,2.$

Then

$$\left[\begin{array}{l} a_{11}^1 a_{22}^2 + a_{11}^1 d_{22}^2 e_{22}^2 \delta_{22}^2 + a_{22}^2 d_{11}^1 e_{11}^1 \delta_{11}^1 + \\ +d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 \delta_{11}^1 \delta_{22}^2 - a_{12}^1 a_{21}^2 - a_{12}^1 d_{21}^2 e_{21}^2 \delta_{21}^2 - \\ -a_{21}^2 d_{12}^1 e_{12}^1 \delta_{12}^1 - d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 \delta_{12}^1 \delta_{21}^2 \le 0; \\ a_{11}^2 a_{22}^1 + a_{11}^2 d_{22}^1 e_{22}^1 \delta_{22}^1 + a_{22}^1 d_{11}^2 e_{11}^2 \delta_{11}^2 + \\ +d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 \delta_{11}^2 \delta_{22}^1 - a_{12}^2 a_{21}^1 - a_{21}^1 d_{12}^2 e_{12}^2 \delta_{12}^2 - \\ -a_{12}^2 d_{21}^1 e_{21}^1 \delta_{21}^1 - d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 \delta_{21}^1 \delta_{12}^2 \le 0. \end{array}\right.$$

By using matrix norm, we have:

$$\begin{cases} a_{11}^1 a_{22}^2 + a_{22}^2 d_{11}^1 e_{11}^1 \| \Delta_1 \| + a_{11}^1 d_{22}^2 e_{22}^2 \| \Delta_2 \| - \\ - d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 \| \Delta_1 \| . \| \Delta_2 \| - a_{12}^1 a_{21}^2 - a_{12}^1 d_{21}^2 e_{21}^2 \| \Delta_2 \| - \\ - a_{21}^2 d_{12}^1 e_{12}^1 \| \Delta_1 \| - d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 \| \Delta_1 \| . \| \Delta_2 \| \le 0; \\ a_{11}^2 a_{22}^1 + a_{11}^2 d_{22}^1 e_{22}^1 \| \Delta_1 \| + a_{22}^1 d_{11}^2 e_{11}^2 \| \Delta_2 \| - \\ - d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 \| \Delta_1 \| . \| \Delta_2 \| - a_{12}^1 a_{21}^2 - a_{21}^1 d_{12}^2 e_{12}^2 \| \Delta_2 \| - \\ - a_{12}^2 d_{21}^1 e_{21}^1 \| \Delta_1 \| - d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 \| \Delta_1 \| . \| \Delta_2 \| \le 0. \end{cases}$$

Equivalently

$$\begin{cases} \left(d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2\right) \| \Delta_1 \| . \| \Delta_2 \| + \\ + \left(a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1\right) \| \Delta_1 \| + \\ + \left(a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2\right) \| \Delta_2 \| - \left(a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2\right) \ge 0; \\ \left(d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 + d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2\right) \| \Delta_1 \| . \| \Delta_2 \| + \\ + \left(a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1\right) \| \Delta_1 \| + \\ + \left(a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2\right) \| \Delta_2 \| - \left(a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2\right) \ge 0. \end{cases}$$

According to the assumption that the switched positive linear system (1) is exponentially stable with $A_1, A_2 \in \mathbb{R}^{2\times 2}$ being Metzler Hurwitz stable matrices, then $a_{11}^1 < 0, a_{22}^1 < 0,$ $a_{11}^2 < 0, a_{22}^2 < 0,$

$$\begin{vmatrix} a_{11}^1 & a_{21}^2 \\ a_{12}^1 & a_{22}^2 \end{vmatrix} > 0, \begin{vmatrix} a_{11}^2 & a_{21}^1 \\ a_{12}^2 & a_{22}^1 \end{vmatrix} > 0,$$

$a_{ij}^k \ge 0, k = 1, 2, \forall i \ne j = 1, 2$ and D_1, D_2, E_1, E_2 are given non-negative definite matrices. We have

$$\begin{aligned} &\left(d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 + d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2\right) \ge 0, \\ &\left(a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1\right) \ge 0, \\ &\left(a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2\right) \ge 0, \quad a_{12}^1 a_{21}^2 - a_{11}^1 a_{22}^2 > 0, \\ &\left(d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1\right) > 0, \\ &\left(a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1\right) > 0, \\ &\left(a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2\right) \ge 0, \quad a_{12}^1 a_{21}^2 - a_{11}^2 a_{22}^1 > 0. \end{aligned}$$

Case 1: Find $r_1 = \min \| \Delta \|_S$ such that $\| \Delta_1 \| \ge 0$, $\| \Delta_2 \| \ge 0$ and

$(d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2) \| \Delta_1 \| . \| \Delta_2 \| +$
$+(a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1) \| \Delta_1 \| +$
$+(a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2) \| \Delta_2 \| - (a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2) \ge 0.$ It is divided into two claims:

$$d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 = 0$$

or $d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 \ne 0$

Claim 1: $d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 + d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 = 0$, we get

$(a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1) \| \Delta_1 \| +$
$+(a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2) \| \Delta_2 \| - (a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2) \ge 0.$

Because $(a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1)$ or $(a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2)$ are not simultaneously zero.

- If $a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1 = 0$ then

$$r_1 = \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2}.$$

- If $a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2 = 0$ then

$$r_1 = \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1}.$$

- If $a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1 \ne 0$ and $a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2 \ne 0$ then

$$r_1 = \min\left\{ \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2}, \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1} \right\}.$$

Claim 2: $d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 + d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 \ne 0$, we get

$$\| \Delta_1 \| \| \Delta_2 \| + \frac{a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1}{d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2} \| \Delta_1 \| + \frac{a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2}{d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2} \| \Delta_2 \| - \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2} \ge 0.$$

Using Lemma 2.2, we have $r_1 = \min\{x+y\}$ such that $2xy + \alpha x + \beta y - \gamma \geq 0,\ x \geq 0,\ \ y \geq 0,$

where $x = \| \Delta_1 \|;\ y = \| \Delta_2 \|;$

$$\frac{\alpha}{2} = \frac{a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1}{d_{12}^1 e_{12}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2} \geq 0;$$

$$\frac{\beta}{2} = \frac{a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2}{d_{12}^1 e_{12}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2} \geq 0;$$

$$\frac{\gamma}{2} = \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{d_{12}^1 e_{12}^1 d_{22}^2 e_{22}^2 + d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2} > 0\,.$$

Case 2: Find $r_2 = \min \| \Delta \|_S$ such that $\| \Delta_1 \| \geq 0, \| \Delta_2 \| \geq 0$ and

$$\left(d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1\right) \| \Delta_1 \| . \| \Delta_2 \| +$$
$$+\left(a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1\right) \| \Delta_1 \| +$$
$$+\left(a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2\right) \| \Delta_2 \| - \left(a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2\right) \geq 0.$$

It is divided into two parts.

Claim 1: $d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 = 0$,we get

$$\left(a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1\right) \| \Delta_1 \| +$$
$$+\left(a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2\right) \| \Delta_2 \| - \left(a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2\right) \geq 0.$$

Because $a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1$ or $a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2$ are not simultaneously zero.

- If $a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1 = 0$ then

$$r_2 = \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2}.$$

- If $a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2 = 0$ then

$$r_2 = \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1}.$$

- If $a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1 \neq 0$ and $a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2 \neq 0$ then

$$r_2 = \min\left\{ \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2}, \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1} \right\}.$$

Claim 2: $d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 \neq 0$, we get

$$\| \Delta_1 \| \| \Delta_2 \| + \frac{a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1}{d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1} \| \Delta_1 \| +$$
$$+ \frac{a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2}{d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1} \| \Delta_2 \| -$$
$$- \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1} \geq 0.$$

Using Lemma 2.2, we have $r_2 = \min\{x+y\}$ such that $2xy + \alpha x + \beta y - \gamma \geq 0,\ x \geq 0,\ \ y \geq 0,$ where, $x = \| \Delta_1 \|;\ \ y = \| \Delta_2 \|;$

$$\frac{\alpha}{2} = \frac{a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1}{d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1} \geq 0;$$

$$\frac{\beta}{2} = \frac{a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2}{d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1} \geq 0,$$

$$\frac{\gamma}{2} = \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1} > 0\,.$$

Combining the above two cases, we come to a formula $r_{\mathbb{R}_+}(\mathcal{A}, \mathcal{D}, \mathcal{E}) = \min\{r_1, r_2\}$.

Example 3.1. Consider the switched positive linear system (1) with N={1, 2}, subjected to affine perturbations of the form (11), where

$$A_1 = \begin{pmatrix} -2.02 & 1.01 & 0.11 \\ 1.13 & -1.03 & 0.12 \\ 0.01 & 0.12 & -2.1 \end{pmatrix},\ D_1 = \begin{pmatrix} 1.21 \\ 0.13 \\ 0.02 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} 0.01 & 1.03 & 1.01 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -1.04 & 1.13 & 0.02 \\ 0.01 & -2.01 & 1.01 \\ 1.01 & 0.12 & -2.05 \end{pmatrix},\ D_2 = \begin{pmatrix} 1.32 \\ 1.21 \\ 0.01 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} 1.02 & 1.05 & 0.03 \end{pmatrix}.$$

It is easy to verify that the Metzler matrices A_k, k=1, 2 are Hurwitz stable and

$$A_0 = \begin{pmatrix} -1.04 & 1.13 & 0.11 \\ 1.13 & -1.03 & 1.01 \\ 1.01 & 0.12 & -2.05 \end{pmatrix}.$$

From Theorem 3.1, we obtain

$$0.1788 \leq r_{\mathbb{R}_+}(\mathcal{A}, \mathcal{D}, \mathcal{E}) \leq 0.2350.$$

Example 3.2. Consider the switched positive linear system (1) with N={1, 2}, subjected to affine perturbations of the form (11), where

$$A_1 = \begin{pmatrix} -0.5 & 0 \\ 0.2 & -0.3 \end{pmatrix}, D_1 = \begin{pmatrix} 0.2 \\ 0.3 \end{pmatrix}, E_1 = \begin{pmatrix} 0.1 & 0.2 \end{pmatrix};$$

$$A_2 = \begin{pmatrix} -0.1 & 0.2 \\ 0 & -0.4 \end{pmatrix}, D_2 = \begin{pmatrix} 0.2 \\ 0.1 \end{pmatrix}, E_2 = \begin{pmatrix} 0.3 & 0.1 \end{pmatrix}.$$

Since $a_{ij}^k \geq 0, \forall i \neq j = 1,2; k = 1,2$ and

$\lambda_1(A_1) = -0.3, \lambda_2(A_1) = -0.5;$

$\lambda_1(A_2) = -0.1, \lambda_2(A_2) = -0.4.$

The switched positive linear system (1) is exponentially stable, because

$$\begin{vmatrix} a_{11}^1 & a_{12}^2 \\ a_{21}^1 & a_{22}^2 \end{vmatrix} = \begin{vmatrix} -0.5 & 0 \\ 0 & -0.4 \end{vmatrix} = 0.2 > 0,$$

$$\begin{vmatrix} a_{11}^2 & a_{12}^1 \\ a_{21}^2 & a_{22}^1 \end{vmatrix} = \begin{vmatrix} -0.1 & 0 \\ 0.2 & -0.4 \end{vmatrix} = 0.04 > 0.$$

We have.

Case 1: $d_{12}^1 e_{12}^1 d_{21}^2 e_{21}^2 + d_{11}^1 e_{11}^1 d_{22}^2 e_{22}^2 = 0,$

$a_{21}^2 d_{12}^1 e_{12}^1 - a_{22}^2 d_{11}^1 e_{11}^1 = 0.08 \neq 0$, $a_{12}^1 d_{21}^2 e_{21}^2 - a_{11}^1 d_{22}^2 e_{22}^2 = 0$

and $a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2 = 0.2$ then

$$r_1 = \frac{a_{11}^1 a_{22}^2 - a_{12}^1 a_{21}^2}{a_{21}^2 d_{12}^1 e_{12}^2 - a_{22}^2 d_{11}^1 e_{11}^1} = \frac{0.2}{0.08} = 2.5.$$

Case 2: $d_{21}^1 e_{21}^1 d_{12}^2 e_{12}^2 + d_{11}^2 e_{11}^2 d_{22}^1 e_{22}^1 = 0,$

$a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1 = 0$, $a_{21}^1 d_{12}^2 e_{12}^2 - a_{22}^1 d_{11}^2 e_{11}^2 = 0.018 \neq 0$

and $a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2 = 0.03$ then

$$r_2 = \frac{a_{11}^2 a_{22}^1 - a_{12}^1 a_{21}^2}{a_{12}^2 d_{21}^1 e_{21}^1 - a_{11}^2 d_{22}^1 e_{22}^1} = \frac{0.03}{0.018} = 1.6667.$$

Using Theorem 3.2, we obtain

$$r_{\mathbb{R}_+}(\mathcal{A}, \mathcal{D}, \mathcal{E}) = \min\{r_1, r_2\} = 1.6667.$$

4. Conclusion

In this paper, based on conditions of exponential stability of the switched positive linear system and the concept of a stability radius related to the structured affine of a matrix of the subsystem, we propose a new approach to studying the robustness of a linear system. In the case of a two-dimensional switched system with two switching signals, we obtained a formula of the stability radius by estimating the positive real stability radius. Some examples are provided for illustrating the result. Our future work is on the formulas of stability radius for multi-dimensional switched positive linear systems with multiple switching signals.

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