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# CLASSIFICATION OF 8-DIMENSIONAL SOLVABLE LIE ALGEBRAS HAVING 7-DIMENSIONAL HEISENBERG NILRADICAL

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**Abstract**

*In this paper, we classify all indecomposable 8-dimensional solvable Lie algebras whose nilradical is the 7-dimensional Heisenberg algebra. Our method is based on the observation that any solvable Lie algebra L can be viewed as an extension of its nilradical N(L), namely, the maximal nilpotent ideal of L. Using standard techniques from Lie theory, we begin with the 7-dimensional Heisenberg algebra and classify all 8-dimensional algebras that admit it as their nilradical.*

**Keywords:** *Heisenberg algebra, Lie algebra, nilradical Lie algebra, solvable Lie algebra.*

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# PHÂN LOẠI ĐẠI SỐ LIE GIẢI ĐƯỢC 8–CHIỀU CÓ CĂN LŨY LINH HEISENBERG 7–CHIỀU

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**Tóm tắt**

*Trong bài báo này, chúng tôi phân loại tất cả các đại số Lie giải được 8-chiều bất khả phân có căn lũy linh Heisenberg 7-chiều. Phương pháp phân loại của chúng tôi là xem đại số Lie giải được L đã biết như là mở rộng của căn lũy linh N(L) của L, tức là ideal lũy linh tối đại của L. Kết hợp với các kỹ thuật cơ bản của lý thuyết Lie, chúng tôi bắt đầu từ một đại số Heisenberg 7-chiều và phân loại tất cả các đại số Lie giải được nhận nó làm căn lũy linh.*

**Từ khóa:** *Căn lũy linh, đại số Lie, đại số Lie giải được, đại số Heisenberg.*

## 1. Introduction

Under the Levi–Malcev Theorem (Levi, 1905; Malcev, 1945), every finite-dimensional Lie algebra $L$ can be decomposed into a semidirect sum of a semisimple Lie subalgebra and a maximal solvable ideal. Hence, the problem of classifying Lie algebras naturally reduces to the classification of semisimple Lie subalgebras and solvable Lie algebras. The classification of semisimple Lie subalgebras has been completely and rigorously established: over the complex field by Cartan (1894) and over the real field by Grantmatcher (1939). Consequently, the classification of Lie algebras over fields of characteristic zero essentially reduces to the classification of solvable Lie algebras.

At present, the classification of seven-dimensional solvable Lie algebras over both the complex and real fields has been completed up to isomorphism (Gong, 1998; Parry, 2007; Hindeleh & Thompson, 2008; Le et al., 2023). However, a complete classification of 8-dimensional solvable Lie algebras over the complex and real fields has not yet been achieved, and this remains an open problem.

Rubin and Winternitz (1993) established a general classification theorem for solvable Lie algebras with a Heisenberg nilradical. In particular, they provided a classification of 8-dimensional solvable Lie algebras with a 5-dimensional Heisenberg nilradical. However, their theorem does not explicitly mention issues of indecomposability and isomorphism.

In this paper, we present the classification of 8-dimensional solvable Lie algebras having 7-dimensional Heisenberg nilradical. We then examine their indecomposability and isomorphism classes, thereby contributing to classifying 8-dimensional solvable Lie algebras having 7-dimensional Heisenberg nilradical over the complex and real fields. Our method is based on the fact that a given solvable Lie algebra $L$ can be considered as an extension of its nilradical $H(3)$, i.e., the maximal nilpotent ideal of $L$. We start from a 7-dimensional nilpotent Lie algebra and classify all 8-dimensional solvable Lie algebras admitting it as their nilradical.

## 2. Preliminaries

From now on, we will use the following notations:

(1) $\mathbb{K}$ denotes the real field or the complex field.

(2) $L$ is the solvable Lie algebras over the field $\mathbb{K}$.

(3) $Mat(r,\mathbb{K})$ is the class of all $r\times r$ matrices over the field $\mathbb{K}$.

(4) $Aut(L)$ is the class of all automorphisms of $L$.

(5) $I_n$ denotes the identity matrix of order $n$.

(6) $E_{ij}$ the 7-square matrix whose only non-zero entry is 1 in row $i$ column $j$.

**Definition 1.** (1) A $(2n+1)$-dimensional Lie algebra $H(n)$ is called a *Heisenberg algebra* if it has a basis $\{P_1,\ldots,P_n,B_1,\ldots,B_n,H\}$ that satisfies the following conditions

$$[P_i, B_j] = \delta_{ij} H, \quad [P_i, H] = [B_i, H] = 0, \quad \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad i, j = 1, \dots, n.$$

(2) Let $L$ be a Lie algebra. We recall its two characteristic series as follows.

(2a) The *derived series* $(DS)$ is

$$L^0 := L \supset L^1 := [L, L] \supset \dots \supset L^k := [L^{k-1}, L^{k-1}] \supset \dots$$

we say that $L$ is *solvable* if $DS$ terminates, i.e. $L^k = 0$ for some $k$.

(2b) The *lower central series* $(LS)$ is

$$L_0 := L \supset L_1 := [L, L] \supset \dots \supset L_k := [L, L_{k-1}] \supset \dots$$

we say that $L$ is *nilpotent* if $LS$ terminates, i.e. $L_k = 0$ for some $k$.

Every solvable Lie algebra $L$ has a uniquely defined maximal nilpotent ideal, called the nilradical $N(L)$. The dimension of the nilradical satisfies

$$r = \dim N(L) \geq \frac{n}{2}, \quad n = \dim L.$$

Any solvable Lie algebra $L$ can be written as the algebraic sum of the nilradical $N(L)$ and a vector subspace $F$, that is, $L = F + N(L)$.

Rubin and Winternitz (1993) gave a classification of n-dimensional solvable Lie algebras with a given Heisenberg nilradical, which we will refer to as Theorem 1. Applying this result, they classified all 8-dimensional solvable Lie algebras with a 5-dimensional Heisenberg nilradical over the complex and real fields. Based on the classification method of Rubin and Winternitz, combined with classification techniques in Lie theory, we continue to classify 8-dimensional solvable Lie algebras with a 7-dimensional Heisenberg nilradical. Our method is to start from a 7-dimensional Heisenberg algebra $H(3)$, then classify all 8-dimensional solvable Lie algebras that have $H(3)$ as their nilradical. With the obtained list of 8-dimensional solvable Lie algebras, we use an algorithm in Maple to eliminate the decomposable Lie algebras and another computer algebra tool which is the so-called *triangular decomposition* (Nguyen et al., 2025) to check for isomorphisms between the remaining indecomposable solvable Lie algebras. We restate this theorem below for convenience and future reference, and then apply it to the present context.

**Theorem 2** (Rubin and Winternitz, 1993). For convenience, we restate Theorem 1 of Rubin and Winternitz (1993) here. Every indecomposable solvable Lie algebra $L(n, f)$ (over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$), containing the Heisenberg algebra $H(n)$ as its nilradical, can be written in a canonical basis $\{S_1, \dots, S_f, P_1, \dots, P_n, B_1, \dots, B_n, H\}$ that satisfies the following conditions

$$\begin{pmatrix}[S_\alpha,H]\\ [S_\alpha,P_1]\\ \cdots\\ [S_\alpha,P_n]\\ [S_\alpha,B_1]\\ \cdots\\ [S_\alpha,B_n]\end{pmatrix}=M_\alpha\begin{pmatrix}H\\ P_1\\ \cdots\\ P_n\\ B_1\\ \cdots\\ B_n\end{pmatrix},\quad M_\alpha=\begin{pmatrix}2\lambda_\alpha & 0\\ 0 & \lambda_\alpha I_{2n}+X_\alpha\end{pmatrix},\quad X_\alpha=\begin{pmatrix}A_\alpha & C_\alpha\\ D_\alpha & -A_\alpha^T\end{pmatrix},$$

$$\begin{aligned}&C_\alpha=C_\alpha^T,\quad D_\alpha=D_\alpha^T,\quad A_\alpha,C_\alpha,D_\alpha\in Mat(n,\mathbb{K}),\\ &[S_\alpha,S_\beta]=r_{\alpha\beta}H,\quad r_{\alpha\beta}\in\mathbb{K},\quad \alpha,\beta=1,\dots,f.\end{aligned}$$

The constants $\lambda_\alpha$ satisfy $\lambda_1=0$ or $\lambda_1=1,\quad \lambda_2=\ldots=\lambda_f=0.$

The matrices $X_\alpha$ satisfy $[X_\alpha,X_\beta]=0,\quad \alpha,\beta=1,\dots,f.$

For $\lambda_1=0$ (respectively, $\lambda_1=1$) the sets $\{X_1,\dots,X_f\}$ (respectively, $\{X_2,\dots,X_f\}$) are linearly nilindependent.

The constants $r_{\alpha\beta}$ satisfy

$$\begin{aligned}&r_{\alpha\beta}=-r_{\beta\alpha}=0\quad\text{for } \lambda_1=1,\\ &r_{\alpha\beta}=-r_{\beta\alpha}\in\mathbb{K}\quad\text{for } \lambda_1=0.\end{aligned}$$

The dimension of the Lie algebra $L(n,f)$ is

$$\dim L(n,f)=2n+1+f\quad 0\le f\le n+1.$$

**3. Construction of Lie algebras**

In the first stage, we construct 8-dimensional solvable Lie algebras $L$ having 7-dimensional Heisenberg nilradical $H(3)$. We assume that the basis of $H(3)$ is $\{P_1,P_2,P_3,B_1,B_2,B_3,H\}$. By adding to the basis $\{P_1,P_2,P_3,B_1,B_2,B_3,H\}$ one element $S$, we obtain that the basis of $L$ is $\{S,P_1,P_2,P_3,B_1,B_2,B_3,H\}$. Then, Lie brackets of $L$ are determined by $[S,H],[S,P_j]$ and $[S,B_j]$ for $j=1,2,3$. Since the derived algebra of a solvable Lie algebra is contained in its nilradical, these Lie brackets can be represented as follows:

$$([S,\xi])=M(\xi),\ \xi^T=(H\quad P_1\quad P_2\quad P_3\quad B_1\quad B_2\quad B_3).$$

Then, $M\in Mat(7,\mathbb{K})$ is called *structure matrix*. To classify the Lie algebra $L,$ we will classify all possible structure matrices $M.$ The following techniques will be used.

(1) First, we use Theorem 2 to give the original form of the matrix $M$ that satisfies the following condition

$$M=\begin{pmatrix}2\lambda & 0\\ 0 & \lambda I_6+X\end{pmatrix},\quad \lambda=0,1,\quad X\in Mat(6,\mathbb{K}).$$

(2) Next, we determine all possible Jordan canonical forms of the matrix $X$.

(3) Finally, we use the following transformations to simplify $X$.

(i) By applying Theorem 2, the entries of $X$ must satisfy

$$X=\begin{pmatrix}A & C\\ D & -A^T\end{pmatrix},\quad C=C^T,\quad D=D^T,\quad A,C,D\in Mat(6,\mathbb{K}).$$

(ii) We use automorphisms group of $Aut(H(3))$ and $Aut(L)$ to select an alternative basis in which the matrices

$$X'=P^{-1}XP,\ M'=Q^{-1}MQ\quad P\in Aut(H(3)),Q\in Aut(L).$$

Then $M'$ is also a structure matrix, and $X'$ is again of the form of the matrix $X$ in Theorem 2.

(iii) We choose another basis for $L$ by:

$$S'=\mu S,\ P_j{}'=\alpha_j P_j,\ B_j{}'=\beta_j B_j,\ H'=\upsilon H,\ j=1,2,3,\ \alpha_j,\beta_j,\mu,\upsilon\in\mathbb{K}.$$

Next, we will classify all 8-dimensional solvable Lie algebras having 7-dimensional Heisenberg nilradical in section 4 below.

## 4. Classification of 8-dimensional solvable Lie algebras having 7-dimensional Heisenberg nilradical

### 4.1. The situation for $\mathbb{K}=\mathbb{C}$

According to Theorem 2, we have the original forms of the structure matrix as follows

$$M=\begin{pmatrix}2\lambda & 0\\ 0 & \lambda I_6+X\end{pmatrix},\quad \lambda=0,1,\quad X\in Mat(6,\mathbb{C}).$$

First, we have the Jordan canonical forms of matrix $X$ in Table 1.

**Table 1: The Jordan canonical form of matrix $X$ over the field $\mathbb{C}$**

| Case | The Jordan canonical form of matrix $X$ |
|---|---|
| $X_1$ | $diag\ \ a,b,c,d,e,f$ |
| $X_2$ | $diag\ \ a,b,c,d,e,e\ \ +E_{56}$ |
| $X_3$ | $diag\ \ a,b,c,c,d,d\ \ +E_{34}+E_{56}$ |
| $X_4$ | $diag\ \ a,a,b,b,c,c\ \ +E_{12}+E_{34}+E_{56}$ |
| $X_5$ | $diag\ \ a,a,a,b,b,b\ \ +E_{12}+E_{23}+E_{45}+E_{56}$ |
| $X_6$ | $diag\ \ a,b,c,c,c,c\ \ +E_{34}+E_{45}+E_{56}$ |
| $X_7$ | $diag\ \ a,a,a,a,b,b\ \ +E_{12}+E_{23}+E_{34}+E_{56}$ |
| $X_8$ | $diag\ \ a,a,a,a,a,a\ \ +E_{12}+E_{23}+E_{34}+E_{45}+E_{56}$ |

We now present the classification of 8-dimensional solvable Lie algebras corresponding to Case $X_1$ in Table 1. Specifically,

$$X = diag\ \ a,b,c,d,e,f\ \ ,\ \ a,b,c,d,e,f \in \mathbb{C}.$$

By Theorem 2, we have

$$X = diag(a,b,c,-a,-b,-c),\quad \begin{cases} \left|c\right| \le \left|b\right| \le \left|a\right| \\ 0 \le arg(a), \arg(b), \arg(c) \le \pi \\ \left|a\right| = \left|b\right| \Rightarrow \arg(b) \le \arg(a) \\ \left|b\right| = \left|c\right| \Rightarrow \arg(c) \le \arg(b). \end{cases}$$

Substitute matrix $X$ into matrix $M$ and we get

$$M = diag(2\lambda, \lambda + a, \lambda + b, \lambda + c, \lambda - a, \lambda - b, \lambda - c).$$

For $\lambda = 1,$ we have

$$M = diag(2, 1 + a, 1 + b, 1 + c, 1 - a, 1 - b, 1 - c).$$

We change the basis of $L$ by the basis $\{2S, 4H, 2P_1, 2P_2, 2P_3, 2B_1, 2B_2, 2B_3\}$. Then, $\{4H, 2P_1, 2P_2, 2P_3, 2B_1, 2B_2, 2B_3\}$ is also a basis of $H(3)$ and $2S$ is also a complementary element. Therefore, we have

$$\begin{bmatrix} [2S,4H] \\ [2S,2P_1] \\ [2S,2P_2] \\ [2S,2P_3] \\ [2S,2B_1] \\ [2S,2B_2] \\ [2S,2B_3] \end{bmatrix} = \begin{pmatrix} 2 &&&&&& \\ & 1+a &&&&& \\ && 1+b &&&& \\ &&& 1+c &&& \\ &&&& 1-a && \\ &&&&& 1-b & \\ &&&&&& 1-c \end{pmatrix} \begin{pmatrix} 4H \\ 2P_1 \\ 2P_2 \\ 2P_3 \\ 2B_1 \\ 2B_2 \\ 2B_3 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} [S,H] \\ [S,P_1] \\ [S,P_2] \\ [S,P_3] \\ [S,B_1] \\ [S,B_2] \\ [S,B_3] \end{pmatrix} = \begin{pmatrix} 1 &&&&&& \\ & a' &&&&& \\ && b' &&&& \\ &&& c' &&& \\ &&&& 1-a' && \\ &&&&& 1-b' & \\ &&&&&& 1-c' \end{pmatrix} \begin{pmatrix} H \\ P_1 \\ P_2 \\ P_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix},\quad \begin{cases} a' = \dfrac{1+a}{2} \\ b' = \dfrac{1+b}{2} \\ c' = \dfrac{1+c}{2}. \end{cases}$$

For simplicity, we denote $a, b, c$ in place of $a', b', c'$. We obtain the Lie algebra:

$$L_{1,1}:\begin{cases} M=(1,a,b,c,1-a,1-b,1-c) \\ |c|\le|b|\le|a| \\ 0\le arg(a),\arg(b),\arg(c)\le\pi \\ |a|=|b|\Rightarrow\arg(b)\le\arg(a) \\ |b|=|c|\Rightarrow\arg(c)\le\arg(b). \end{cases}$$

For $\lambda=0,$ we have

$$M=diag(0,a,b,c,-a,-b,-c).$$

If $a=0$ then $a=b=c=0$, implies $M$ is zero matrix. We obtain the Lie algebra:

$$L_{1,0,1}:M=diag(0,0,0,0,0,0,0).$$

If $a\ne 0$ then $\{aS,a^2H,aP_1,aP_2,aP_3,aB_1,aB_2,aB_3\}$ is also a basis of $L$. Moreover, $\{a^2H,aP_1,aP_2,aP_3,aB_1,aB_2,aB_3\}$ is also a basis of $H(3)$ and $aS$ is also a complementary element. Therefore, we have

$$\begin{bmatrix}[aS,a^2H]\\ [aS,aP_1]\\ [aS,aP_2]\\ [aS,aP_3]\\ [aS,aB_1]\\ [aS,aB_2]\\ [aS,aB_3]\end{bmatrix}=\begin{pmatrix}0&&&&&&\\ &a&&&&&\\ &&b&&&&\\ &&&c&&&\\ &&&&-a&&\\ &&&&&-b&\\ &&&&&&-c\end{pmatrix}\begin{pmatrix}a^2H\\ aP_1\\ aP_2\\ aP_3\\ aB_1\\ aB_2\\ aB_3\end{pmatrix}.$$

This implies that

$$\begin{pmatrix}[S,H]\\ [S,P_1]\\ [S,P_2]\\ [S,P_3]\\ [S,B_1]\\ [S,B_2]\\ [S,B_3]\end{pmatrix}=\begin{pmatrix}0&&&&&&\\ &1&&&&&\\ &&a'&&&&\\ &&&b'&&&\\ &&&&-1&&\\ &&&&&-a'&\\ &&&&&&-b'\end{pmatrix}\begin{pmatrix}H\\ P_1\\ P_2\\ P_3\\ B_1\\ B_2\\ B_3\end{pmatrix},\quad \begin{cases}a'=\dfrac{b}{a}\\ b'=\dfrac{c}{a}.\end{cases}$$

For simplicity, we denote $a,b$ in place of $a',b'$, we obtain the Lie algebra:

$$L_{1,0,2}:\begin{cases} M=diag(0,1,a,b,-1,a,b) \\ |b|\le|a|\le 1 \\ 0\le\arg(a),\arg(b)\le\pi \\ |a|=|b|\Rightarrow\arg(b)\le\arg(a). \end{cases}$$

Similarly, we achieve the classification of all 8-dimensional solvable Lie algebras whose nilradical is the 7-dimensional Heisenberg algebra over $\mathbb{C}$.

Next, we use an algorithm in Maple to eliminate the decomposable Lie algebras and use another computer algebra tool which is the so-called *triangular decomposition* (Nguyen et al., 2025), to check for isomorphisms between the remaining indecomposable solvable Lie algebras. Table 2 below presents a complete classification of all indecomposable solvable Lie algebras up to isomorphism.

**Table 2: Classification of all 8-dimensional solvable Lie algebras $L$ having 7-dimensional Heisenberg nilradical over $\mathbb{C}$**

| Case | Matrix $M$ | Notes |
|---|---|---|
| $L_{1,1}$ | $diag(1,a,b,c,1-a,1-b,1-c)$ | $\lvert c\rvert \le \lvert b\rvert \le \lvert a\rvert$<br>$0 \le arg(a), \arg(b), \arg(c) \le \pi$<br>$\lvert a\rvert = \lvert b\rvert \Rightarrow \arg(b) \le \arg(a)$<br>$\lvert b\rvert = \lvert c\rvert \Rightarrow \arg(c) \le \arg(b)$ |
| $L_{2,1}$ | $diag(2,a,b,1,2-a,2-b,1)+E_{47}$ | $\lvert b\rvert \le \lvert a\rvert$<br>$0 \le \arg(a), \arg(b) \le \pi$<br>$\lvert a\rvert = \lvert b\rvert \Rightarrow \arg(b) \le \arg(a)$ |
| $L_{3,1}$ | $diag(1,a,b,b,1-a,1-b,1-b)+E_{34}-E_{76}$ | $0 \le \arg(a),\ \arg(b) \le \pi$ |
| $L_{4,1}$ | $diag(2,a,a,1,2-a,2-a,1)+E_{23}+E_{46}$ | $0 \le \arg(a) \le \pi$ |
| $L_{5,1}$ | $diag(1,a,a,a,1-a,1-a,1-a)$<br>$+E_{23}+E_{34}-E_{65}-E_{76}$ | $0 \le \arg(a) \le \pi$ |
| $L_{6,1}$ | $diag(2,a,1,1,2-a,1,1)+E_{34}+E_{47}-E_{76}$ | $0 \le \arg(a) \le \pi$ |
| $L_{7,1}$ | $diag(2,1,1,1,1,1,1)+E_{23}+E_{36}+E_{47}-E_{65}$ | |
| $L_{8,1}$ | $diag(2,1,1,1,1,1,1)+E_{23}+E_{34}+E_{47}-E_{65}-E_{76}$ | |

### 4.2. The situation for $\mathbb{K}=\mathbb{R}$

While the special cases over $\mathbb{C}$ have been thoroughly classified, the case $\mathbb{R}$ gives rise to additional exceptional structures that do not appear in the complex setting. Table 3 presents the distinct Jordan canonical forms of the matrix $X$ that occur over the field $\mathbb{R}$.

**Table 3: The Jordan canonical form of matrix $X$ over the field $\mathbb{R}$.**

| Case | The Jordan canonical form of matrix $X$ |
|---|---|
| $X^9$ | $diag(a,b,c,d,e,e)+f(E_{56}-E_{65})$ |
| $X^{10}$ | $diag(a,b,c,c,d,d)+E_{34}+e(E_{56}-E_{65})$ |
| $X^{11}$ | $diag(a,a,b,b,c,c)+E_{12}+E_{34}+d(E_{56}-E_{65})$ |
| $X^{12}$ | $diag\ \ a,a,a,a,b,b\ \ +E_{12}+E_{23}+E_{34}+c(E_{56}-E_{65})$ |

| | |
|---|---|
| $X^{13}$ | $diag(a,b,c,c,e,e)+d(E_{34}-E_{43})+f(E_{56}-E_{65})$ |
| $X^{14}$ | $diag(a,a,b,b,d,d)+E_{12}+c(E_{34}-E_{43})+e(E_{56}-E_{65})$ |
| $X^{15}$ | $diag(a,a,c,c,e,e)+b(E_{12}-E_{21})+d(E_{34}-E_{43})+f(E_{56}-E_{65})$ |
| $X^{16}$ | $diag(a,b,c,c,e,e)+E_{35}+E_{46}+d(E_{34}-E_{43})+f(E_{56}-E_{65})$ |
| $X^{17}$ | $diag(a,a,b,b,d,d)+E_{12}+E_{35}+E_{46}+c(E_{34}-E_{43})+e(E_{56}-E_{65})$ |
| $X^{18}$ | $diag(a,a,c,c,e,e)+E_{35}+E_{46}+b(E_{12}-E_{21})+d(E_{34}-E_{43})+f(E_{56}-E_{65})$ |

Now, We present the classification of 8-dimensional solvable Lie algebras corresponding to Case $X^9$ in Table 3. Specifically,

$$X=\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & c & & & \\ & & & d & & \\ & & & & e & f \\ & & & & -f & e \end{pmatrix},\ a,b,c,d,e,f\in\mathbb{R}.$$

We have

$$P^{-1}XP=\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & e & & & f \\ & & & d & & \\ & & & & c & \\ & & -f & & & e \end{pmatrix},\ P=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & 1 & \\ & & & 1 & & \\ & & 1 & & 0 & \\ & & & & & 1 \end{pmatrix}.$$

By Theorem 2, we have

$$X=\begin{pmatrix} a & & & & & \\ & b & & & & \\ & & 0 & & & f \\ & & & -a & & \\ & & & & -b & \\ & & -f & & & 0 \end{pmatrix},\ 0\le b\le a,\ 0<f.$$

Substitute matrix $X$ into matrix $M$ and we get

$$M=\begin{pmatrix} 2\lambda & & & & & & \\ & \lambda+a & & & & & \\ & & \lambda+b & & & & \\ & & & \lambda & & & f \\ & & & & \lambda-a & & \\ & & & & & \lambda-b & \\ & & & -f & & & \lambda \end{pmatrix}.$$

For $\lambda=1,$ we have

$$M=\begin{pmatrix} 2 & & & & & & \\ & 1+a & & & & & \\ & & 1+b & & & & \\ & & & 1 & & & f \\ & & & & 1-a & & \\ & & & & & 1-b & \\ & & & -f & & & 1 \end{pmatrix}.$$

We have

$$P^{-1}MP=\begin{pmatrix} 2 & & & & & & \\ & 1+a & & & & & \\ & & 1+b & & & & \\ & & & 1 & & & 1 \\ & & & & 1-a & & \\ & & & & & 1-b & \\ & & & -f^2 & & & 1 \end{pmatrix}, P=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & f & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{pmatrix}.$$

For simplicity, we denote $a,b,c$ in place of $1+a,1+b,f^2$, we obtain the Lie algebra:

$$L^{9,1}:\begin{cases} M=diag(2,a,b,1,2-a,2-b,1)+E_{47}-cE_{74} \\ 1\le b\le a,\ \ 0<c. \end{cases}$$

For $\lambda=0,$ we have

$$M = \begin{pmatrix} 0 & & & & & & \\ & a & & & & & \\ & & b & & & & \\ & & & 0 & & & f \\ & & & & -a & & \\ & & & & & -b & \\ & & & -f & & & 0 \end{pmatrix}.$$

If $a = 0$ then $a = b = c = 0$, implies

$$M = diag(0,0,0,0,0,0,0) + f\ \ E_{47} - E_{74}\ \ .$$

We change the basis of $L$ by the basis $\{fS, f^2H, fP_1, fP_2, fP_3, fB_1, fB_2, fB_3\}$. Then, $\{f^2H, fP_1, fP_2, fP_3, fB_1, fB_2, fB_3\}$ is also a basis of $H(3)$ and $fS$ is also a complementary element. Therefore, we have

$$\begin{bmatrix} [fS, f^2H] \\ [fS, fP_1] \\ [fS, fP_2] \\ [fS, fP_3] \\ [fS, fB_1] \\ [fS, fB_2] \\ [fS, fB_3] \end{bmatrix} = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & f \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & -f & & & 0 \end{pmatrix} \begin{pmatrix} f^2H \\ fP_1 \\ fP_2 \\ fP_3 \\ fB_1 \\ fB_2 \\ fB_3 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} [S, H] \\ [S, P_1] \\ [S, P_2] \\ [S, P_3] \\ [S, B_1] \\ [S, B_2] \\ [S, B_3] \end{pmatrix} = \begin{pmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & 1 \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & -1 & & & 0 \end{pmatrix} \begin{pmatrix} H \\ P_1 \\ P_2 \\ P_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}.$$

It creates the following Lie algebras:

$$L^{9,0,1} : M = diag(0,0,0,0,0,0,0) + E_{47} - E_{74}.$$

If $a \neq 0$, we have

$$P^{-1}MP=\begin{pmatrix}0&&&&&&\\&a&&&&&\\&&b&&&&\\&&&0&&&a\\&&&&-a&&\\&&&&&-b&\\&&&\frac{-f^2}{a}&&&0\end{pmatrix},\ P=\begin{pmatrix}1&&&&&\\&1&&&&\\&&\frac{f}{a}&&&\\&&&1&&\\&&&&1&\\&&&&&1\end{pmatrix}.$$

We change the basis of $L$ by the basis $\{aS,a^2H,aP_1,aP_2,aP_3,aB_1,aB_2,aB_3\}$. Then, $\{a^2H,aP_1,aP_2,aP_3,aB_1,aB_2,aB_3\}$ is also a basis of $H(3)$ and $aS$ is also a complementary element. Therefore, we have

$$\begin{bmatrix}[aS,a^2H]\\ [aS,aP_1]\\ [aS,aP_2]\\ [aS,aP_3]\\ [aS,aB_1]\\ [aS,aB_2]\\ [aS,aB_3]\end{bmatrix}=\begin{pmatrix}0&&&&&&\\&a&&&&&\\&&b&&&&\\&&&0&&&a\\&&&&-a&&\\&&&&&-b&\\&&&\frac{-f^2}{a}&&&0\end{pmatrix}\begin{pmatrix}a^2H\\aP_1\\aP_2\\aP_3\\aB_1\\aB_2\\aB_3\end{pmatrix}.$$

This implies that

$$\begin{pmatrix}[S,H]\\ [S,P_1]\\ [S,P_2]\\ [S,P_3]\\ [S,B_1]\\ [S,B_2]\\ [S,B_3]\end{pmatrix}=\begin{pmatrix}0&&&&&&\\&1&&&&&\\&&a'&&&&\\&&&0&&&1\\&&&&-1&&\\&&&&&-a'&\\&&&-b'&&&0\end{pmatrix}\begin{pmatrix}H\\P_1\\P_2\\P_3\\B_1\\B_2\\B_3\end{pmatrix},\quad \begin{cases}a'=\dfrac{b}{a}\\ b'=\dfrac{f^2}{a^2}.\end{cases}$$

For simplicity, we denote $a,b$ in place of $a',b'$, we obtain the Lie algebra:

$$L^{9,0,2}:\begin{cases}M=diag(0,1,a,0,-1,-a,0)+E_{47}-bE_{74}\\ 0\le a\le 1,\ 0<b.\end{cases}$$

Similarly, we achieve the classification table of all 8-dimensional solvable Lie algebras having 7-dimensional Heisenberg nilradical over $\mathbb{R}$.

Next, we use algorithm in Maple to eliminate the decomposable Lie algebras and use the triangular decomposition algorithm (Nguyen et al., 2025) to check for isomorphisms between the remaining indecomposable solvable Lie algebras. Table 4 below presents a complete classification of all indecomposable solvable Lie algebras up to isomorphism.

**Table 4: Classification of all 8-dimensional solvable Lie algebras $L$ having 7-dimensional Heisenberg nilradical over $\mathbb{R}$**

| Case | Matrix $M$ | Notes |
|---|---|---|
| $L^{1,1}$ | $(1,a,b,c,1-a,1-b,1-c)$ | $\frac{1}{2} \le c \le b \le a$ |
| $L^{2,1}$ | $(2,a,b,1,2-a,2-b,1)+E_{47}$ | $1 \le b \le a$ |
| $L^{3,1}$ | $(1,a,b,b,1-a,1-b,1-b)+E_{34}-E_{76}$ | $\frac{1}{2} \le b \le a$ |
| $L^{4,1}$ | $(2,a,a,1,2-a,2-a,1)+E_{23}+E_{46}$ | $1 \le a$ |
| $L^{5,1}$ | $(1,a,a,a,1-a,1-a,1-a)+E_{23}+E_{34}-E_{65}-E_{76}$ | $\frac{1}{2} \le a$ |
| $L^{6,1}$ | $(2,1+a,1,1,1-a,1,1)+E_{34}+E_{47}-E_{76}$ | $0 \le a$ |
| $L^{7,1}$ | $(2,1,1,1,1,1,1)+E_{23}+E_{36}+E_{47}-E_{65}$ | |
| $L^{8,1}$ | $(2,1,1,1,1,1,1)+E_{23}+E_{34}+E_{47}-E_{65}-E_{76}$ | |
| $L^{9,1}$ | $(2,a,b,1,1-a,1-b,1)+E_{47}-cE_{74}$ | $1 \le b \le a,\ 0 < c$ |
| $L^{10,1}$ | $(2,a,1,1,2-a,1,1)+E_{36}+E_{47}-bE_{74}$ | $1 \le a,\ 0 < b$ |
| $L^{11,1}$ | $(2,a,a,1,2-a,2-a,1)+E_{47}-bE_{74}$ | $1 \le a,\ 0 < b$ |
| $L^{12,1}$ | $(2,1,1,1,1,1,1)+E_{23}+E_{36}+E_{47}-E_{65}-aE_{74}$ | $0 < a$ |
| $L^{13,1}$ | $(2,a,1,1,2-a,1,1)+E_{36}+E_{47}-bE_{63}-cE_{74}$ | $1 \le a,\ 0 < c \le b$ |
| $L^{14,1}$ | $(2,1,1,1,1,1,1)+E_{25}+E_{36}+E_{47}-aE_{63}-bE_{74}$ | $0 < b \le a$ |
| $L^{15,1}$ | $(2,1,1,1,1,1,1)+E_{25}+E_{36}+E_{47}-aE_{52}-bE_{63}-cE_{74}$ | $0 < c \le b \le a$ |
| $L^{16,1}$ | $(1,a,b,b,1-a,1-b,1-b)$ <br> $+E_{36}+E_{47}+E_{34}+E_{76}-cE_{43}-cE_{67}$ | $\frac{1}{2} \le b \le a,\ 0 < c$ |
| $L^{17,1}$ | $(2,a,a,1,2-a,2-a,1)+E_{23}+E_{25}+E_{36}$ <br> $+E_{47}+E_{65}-bE_{32}-bE_{56}$ | $0 \le a,\ 0 < b$ |
| $L^{18,1}$ | $(2,a,a,1,2-a,2-a,1)+E_{23}+E_{25}+E_{36}$ <br> $+E_{47}-E_{65}+bE_{56}-bE_{32}-cE_{74}$ | $0 \le a,\ 0 < b,c$ |

By combining the results from Tables 2 and 4, we obtain the following theorem.

**Theorem 3.** The class of 8-dimensional solvable Lie algebras having 7-dimensional Heisenberg nilradical consists of 8 families of complex indecomposable solvable Lie algebras and 18 families of real ones.

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