DONG THAP UNIVERSITY JOURNAL OF SCIENCE
Tạp chí Khoa học Đại học Đồng Tháp
Natural Sciences Issue
ISSN 0866-7675 | e-ISSN 2815-567X

DOI: https://doi.org/10.52714/dthu.ns.2670.1925

STABLE STRONG DUALITY FOR COMPOSED OPTIMIZATION PROBLEMS WITH COMPOSED CONSTRAINTS

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Article history

Received: 13/8/2025; Received in revised form: 26/10/2025; Accepted: 03/11/2025

Abstract

This paper studies stable strong duality for composed convex optimization problems with composed constraints. To this end, we establish generalized Farkas-type results and provide characterizations for inequality systems involving composed convex functions under composed constraints. These results extend several existing results in convex and nonconvex programming. In particular, we introduce a sufficient condition ensuring generalized Farkas-type results in the composite setting. The obtained results are then applied to derive strong and stable strong duality for composed convex optimization problems with composed constraints. As illustrations of the results, we present some consequences and applications to special cases.

Keywords: *composed convex optimization problems, DC programs, Farkas-type results, stable strong duality.*

Cite: Dang, H. L., & Tran, H. M. (2026). Stable strong duality for composed optimization problems with composed constraints. *Dong Thap University Journal of Science - Natural Sciences Issue*, *Online First*. https://doi.org/10.52714/dthu.ns.2670.1925

ĐỐI NGẪU MẠNH ỔN ĐỊNH CHO BÀI TOÁN TỐI ƯU HỢP VỚI RÀNG BUỘC HỢP

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Lịch sử bài báo

Ngày nhận: 13/8/2025; Ngày nhận chỉnh sửa: 26/10/2025; Ngày duyệt đăng: 03/11/2025

Tóm tắt

Bài báo này tập trung nghiên cứu tính đối ngẫu mạnh ổn định cho các bài toán tối ưu lồi có hàm mục tiêu và ràng buộc dạng hàm hợp. Để đạt được mục tiêu này, chúng tôi thiết lập các kết quả dạng Farkas tổng quát và đưa ra các đặc trưng cho hệ bất đẳng thức liên quan đến hàm hợp lồi dưới ràng buộc dạng hàm hợp. Những kết quả này mở rộng một số kết quả đã có trong quy hoạch lồi và không lồi. Đặc biệt, chúng tôi đề xuất một điều kiện đủ để đảm bảo các kết quả dạng Farkas tổng quát cho hàm hợp. Các kết quả thu được sau đó được áp dụng để suy ra tính đối ngẫu mạnh và đối ngẫu mạnh ổn định cho các bài toán tối ưu lồi có hàm mục tiêu và ràng buộc dạng hàm hợp. Để minh họa cho các kết quả, chúng tôi trình bày một số hệ quả và ứng dụng cho các trường hợp đặc biệt.

Từ khóa: *bài toán DC, bài toán tối ưu lồi dạng hàm hợp, các kết quả dạng Farkas, đối ngẫu mạnh ổn định.*

1. Introduction

It is well known that duality theory plays an important role in optimization. For a given primal problem, there are different ways to define its dual problems (Bot, 2010; Bot et al., 2003; Feizollahi et al., 2017; Huang et al., 2003; Long et al., 2010), particularly in the context of convex and DC (difference of two convex functions) optimization problems. Duality refers to the situation where the optimal values of the primal problem and that of its dual problem are equal, often under regularity or constraint qualification conditions.

In recent decades, many attempts have been made to study the duality for various classes of optimization problems (Bot, 2010; Bot et al., 2006a, 2006b, 2003; Dinh et al., 2008, 2010, 2014; Feizollahi et al., 2017; Huang et al., 2003; Jeyakumar et al., 2009; Li, 1995; Li et al., 2008; Long et al., 2010; and so on). The study of DC optimization problems dates back to the early developments of convex analysis and functional analysis (Zălinescu, 2002). Fundamental duality principles such as the Fenchel–Rockafellar theorem and the Moreau–Rockafellar theorem formed the basis for later generalizations. Over the past decades, a series of works (Bot, 2010; Bot et al., 2003, 2006a, 2006b; Huang et al., 2003; Jeyakumar et al., 2009; Li, 1995; Li & Ng, 2008) has extended these results to broader convex settings, introducing augmented Lagrangian duality, generalized regularity, and stability concepts that guarantee zero duality gaps in convex and semi-definite programs.

A major advance in the theory came from the introduction of composed convex functions–functions of the form $(f \circ F)(x)$, with f convex and lower semicontinuous, and F a linear or nonlinear continuous mapping. This line of research was initiated and systematically developed by Bot and collaborators (Bot, 2010; Bot et al., 2003, 2006a, 2006b, 2009, 2006c). In particular, Bot et al. (2003) provided early results on duality in the presence of composed convex functions with applications in location theory, while Bot et al. (2009) generalized the Moreau–Rockafellar theorem to composed convex functions, providing a deeper subdifferential and conjugate structure. Building upon conjugate duality, Bot et al. (2006a, 2006b) derived Farkas-type results for inequality systems involving composed convex functions, and established duality under generalized constraint qualification conditions. These results were further refined by Bot et al. (2006c) and Huang et al. (2003) to cover infinite-dimensional spaces under weaker regularity assumptions, thus significantly broadening the applicability of composed convex duality.

Parallel to the above developments, another research stream focused on nonconvex or DC (difference of convex functions) programs and generalized convex systems. Dinh and collaborators (Dinh et al., 2008, 2010, 2014) proposed new Farkas-type results, closedness conditions, and alternative theorems, which extended classical duality beyond convexity. These results demonstrated that stable dual representations can still hold even when convexity fails. Later, Dinh and Mo (2012) further introduced qualification conditions and Farkas-type theorems for systems involving composite functions, unifying the treatment of composed and DC optimization structures.

Despite these advances, there remains a gap in the literature regarding unified duality frameworks that can handle composed convex optimization problems with composed constraints, particularly when the objective itself is a composition such as $(f \circ F)(x) + g(x)$, and the constraint set involves a composition like $(H \circ K)(x) \in -T_+$. The current paper addresses this gap by proposing a generalized problem of the form:

$$\text{(CCP)} \qquad \inf_{(H \circ K)(x) \in -T_+} [(f \circ F)(x) + g(x)],$$

where the spaces involved are locally convex Hausdorff topological vector spaces, and the mappings F, H, K are allowed to take values in extended spaces. The above model encapsulates many previously studied problems, including classical convex programs, composed convex problems as in the works of Bot (2010), Bot et al. (2006a, 2003) and the references therein.

The novelty of our approach lies in the integration of composed mappings in both the objective and the constraints, while still enabling the use of conjugate duality. This leads to the definition of a new dual problem—the Fenchel-Lagrange dual:

$$\text{(CCD)} \qquad \sup_{\substack{(\gamma,\lambda)\in \mathrm{dom} f^*\times T_+^* \\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left\{\inf_{x\in X}[\gamma F(x)+g(x)+\kappa K(x)]-f^*(\gamma)-(\lambda H)^*(\kappa)\right\}.$$

This formulation generalizes the dual constructions proposed by Bot et al. (2006a) and Long et al. (2010), and encompasses several duality principles under a unified structure. By incorporating the conjugates f^* and $(\lambda H)^*$, our framework captures the dual behavior of composed convex systems in both finite and infinite-dimensional settings.

The main contributions of this paper are as follows:

- We develop characterizations of Farkas-type results for systems involving convex composite functions with composed constraints. These characterizations provide both necessary and sufficient conditions, extending results of Bot et al. (2006a), Dinh et al. (2010, 2014) and unifying them within a broader framework.
- Our theoretical findings are applied to establish stable strong duality (in the sense of Jeyakumar et al. (2009)) and strong duality for the primal-dual pair (CCP)–(CCD). In particular, we provide duality results for convex composite problems involving DC structures, thereby generalizing and extending the results established by Dinh et al. (2008, 2010) and Toland (1978).

This paper provides the first comprehensive treatment of convex optimization problems with both composite objectives and composed constraints in a general topological vector space setting, building upon and significantly extending the frameworks in the works of Bot (2010), Bot et al. (2003, 2006), Dinh et al. (2010), Jeyakumar et al. (2009), and Long et al. (2010).

The remainder of the paper is organized as follows. In Section 2, we introduce necessary notation and preliminary results from convex analysis. Section 3 presents a series of Farkas-type theorems for composite systems, including a new sufficient condition. In Section 4, we apply these results to establish stable strong duality for the primal-dual pair (CCP)–(CCD). Section 5 is devoted to illustrating the applicability of the results by treating special cases, such as DC programs with composed constraints and min–max problems.

2. Preliminaries

Consider the extended real line $\overline{\mathbb{R}} := \mathbb{R}\cup\{\infty\}$ with the following conventions: $\infty+\alpha=\alpha+\infty=\infty$ for all $\alpha\in\overline{\mathbb{R}}$, and $\alpha\infty=\infty$ for all $\alpha\in\mathbb{R}_+\cup\{\infty\}$. We also extend the usual order $\leq$ in the real numbers set with $\alpha\leq\infty$ for all $\alpha\in\overline{\mathbb{R}}$.

Let X be a locally convex Hausdorff topological vector space (lcHtvs, in brief) whose topological dual space is denote by X^* and endowed with the weak*-topology. For a set $D\subset X$, the *indicator function* of D, i_D, is defined by

$$i_D(x):=\begin{cases}0, & \text{if } x\in D,\\ \infty, & \text{if } x\notin D.\end{cases}$$

The closure of a set $A\subset X$ will be denoted by $\overline{A}$ and the set A is said to be closed regarding the set $B\subset X$ if $A\cap B=\overline{A}\cap B$ (Bot, 2010).

Let $h: X\to\overline{\mathbb{R}}$. The *domain* and the *epigraph* of h are

$$\begin{aligned}\mathrm{dom}h &:= \{x\in X \mid h(x)\neq\infty\},\\ \mathrm{epi}h &:= \{(x,r)\in X\times\mathbb{R} \mid x\in\mathrm{dom}h,\ h(x)\leq r\}.\end{aligned} \tag{1}$$

We say that h is *proper* if $\mathrm{dom}h \neq \emptyset$; that h is *convex* if $\mathrm{epi}h$ is convex; that h is *lower semi-continuous* (briefly, lsc) if $\mathrm{epi}h$ is a closed subset of $X \times \mathbb{R}$. The set of all proper convex and lsc functions on X will be denoted by $\Gamma(X)$.

For a proper function h, the *conjugate function* of h, $h^*: X^* \to \overline{\mathbb{R}}$, is defined by

$$h^*(x^*) = \sup\{\langle x^*, x\rangle - h(x) \mid x \in \mathrm{dom}\ h\}, \quad \forall x^* \in X^*.$$

The domain and the epigraph of h^* are defined in a similar way as in (1). It is easy to see that

$$h^*(x^*) + h(\bar{x}) - \langle x^*, \bar{x}\rangle \geq 0. \tag{2}$$

For $\bar{x} \in \mathrm{dom}h$ and $\epsilon \geq 0$, the *subdifferential* and the *ϵ-subgradient* of h at $\bar{x}$ are

$$\partial h(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq h(x) - h(\bar{x}),\ \forall x \in X\},$$
$$\partial_\epsilon h(\bar{x}) = \{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \leq h(x) - h(\bar{x}) + \epsilon,\ \forall x \in X\}.$$

Lemma 1. (Zalinescu, 2002) *Let $h: X \to \overline{\mathbb{R}}$ be a proper function, $\bar{x} \in \mathrm{dom}h$ and $\epsilon \geq 0$.*

(a) *If $0 \leq \epsilon_1 \leq \epsilon_2$ then $\partial h(\bar{x}) = \partial_0 h(\bar{x}) \subset \partial_{\epsilon_1} h(\bar{x}) \subset \partial_{\epsilon_2} h(\bar{x})$.*

(b) $x^* \in \partial h(\bar{x})$ if and only if $h^*(x^*) + h(\bar{x}) - \langle x^*, \bar{x}\rangle = 0$.

(c) $x^* \in \partial_\epsilon h(\bar{x})$ if and only if $h^*(x^*) + h(\bar{x}) \leq \langle x^*, \bar{x}\rangle + \epsilon$.

Theorem 1. (Toland, 1978) *Let $h \in \Gamma(X)$ and $g: X \to \overline{\mathbb{R}}$ be an arbitrary function. Then*

$$\inf_{x \in X}[g(x) - h(x)] = \inf_{x^* \in X^*}[h^*(x^*) - g^*(x^*)].$$

Consider an other lcHtvs Y which is partially ordered by a convex cone Y_+ containing the origin of Y ($0_Y \in Y_+$), i.e., $y_1 \leq_Y y_2$ if $y_1 - y_2 \in -Y_+$. We add to Y a greatest element with respect to $\leq_Y$, denoted by ∞_Y. In the enlarged space $Y^\bullet := Y \cup \{\infty_Y\}$, we assume, by convention, $y \leq_Y \infty_Y$ for every $y \in Y^\bullet$ and we also adopt the following conventions concerning the operations in $Y^\bullet$:

$$y + \infty_Y = \infty_Y + y = \infty_Y,\ \forall y \in Y^\bullet, \qquad \alpha\infty_Y = \infty_Y,\ \forall \alpha \geq 0.$$

The *dual cone* of Y_+ is defined by $Y_+^* := \{\gamma \in Y^* \mid \langle \gamma, y\rangle \geq 0,\ \forall y \in Y_+\}$.

Let $F: X \to Y^\bullet$ be a vector-valued function. We call *domain* of F the set

$$\mathrm{dom}F = \{x \in X \mid F(x) \in Y\}$$

and we say that F is proper if $\mathrm{dom}F \neq \emptyset$. The mapping F is said to be *Y_+-convex*, if for all $x, y \in X$ and $t \in]0,1[$ one has $F(tx + (1-t)y) - tF(x) - (1-t)F(y) \in -Y_+$, or, equivalently, $\mathrm{epi}_Y F$ is a convex subset of $X \times Y$ where

$$\mathrm{epi}_Y F := \{(x, y) \in X \times Y : F(x) \in y - Y^+\}.$$

Assume that X_+ is a cone in X, F is said to be $X_+ - Y_+$ *increasing* if $F(x_1) \leq_Y F(x_2)$ whenever $x_1 \leq_X x_2$.

Let further $f: Y \to \overline{\mathbb{R}}$ be a proper function. The composed function $f \circ F: X \to \overline{\mathbb{R}}$ is defined by

$$(f \circ F)(x) := \begin{cases} f(F(x)), & \text{if } x \in \mathrm{dom}F, \\ \infty, & \text{else.} \end{cases}$$

We say that f is *Y_+-increasing* if, for all $y_1, y_2 \in Y$ such that $y_1 \leq_Y y_2$ one always gets $f(y_1) \leq f(y_2)$. Observing that if f is Y_+-increasing, convex and F is Y_+-convex then $f \circ F$ is a convex function.

3. Generalized composite convex Farkas-type results

Let X, Y, Z, T be locally convex Hausdorff topological vector spaces, X^*, Y^*, Z^*, and T^* denote their topological dual spaces (respectively), endowed with the weak*-topology. Let $F: X \to Y^\bullet$, $f: Y \to \overline{\mathbb{R}}$, $g: X \to \overline{\mathbb{R}}$, $K: X \to Z^\bullet$, and $H: Z \to T^\bullet$ be proper mappings satisfying

$$\mathrm{dom}(f \circ F + g) \cap (H \circ K)^{-1}(-T_+) \neq \emptyset, \tag{3}$$

where T_+ is a closed convex cone in T whose dual cone is denoted by T_+^*.

Let Y_+, Z_+ be closed convex cones in Y and Z, respectively. Throughout this section, assume that g is convex; f is convex and Y_+-increasing; F is Y_+-convex; K is Z_+-convex; H is T_+-convex and $Z_+ - T_+$-increasing.

Let $\emptyset \neq \mathcal{V} \subset X^*$. For each $(x^*, \alpha) \in \mathcal{V} \times \mathbb{R}$, we are concerned with the implication

$$(\mathbf{p}_\alpha^{x^*}) \qquad (H \circ K)(x) \in -T_+ \;\Rightarrow\; (f \circ F)(x) + g(x) - \langle x^*, x \rangle \geq \alpha,$$

and, in particular, the implication

$$(\mathbf{p}_\alpha) \equiv (\mathbf{p}_\alpha^{0_{X^*}}) \quad (H \circ K)(x) \in -T_+ \;\Rightarrow\; (f \circ F)(x) + g(x) \geq \alpha.$$

Each pair of $(\mathbf{p}_\alpha)$ and an its equivalent assertion will be called a *generalized composite convex Farkas-type result* (*GCCFR*, in brief) while we will call the statement

$$[(\mathbf{p}_\alpha^{x^*}) \Leftrightarrow (\widetilde{\mathbf{p}}_\alpha^{x^*})], \quad \forall (x^*, \alpha) \in \mathcal{V} \times \mathbb{R},$$

for some $(\widetilde{\mathbf{p}}_\alpha^{x^*})$, a $\mathcal{V}$*-stable GCCFR*. This section aims to establish some necessary and sufficient conditions (in other words, characterizations) or sufficient conditions for GCCFR/ $\mathcal{V}$-stable GCCFR.

To simplify writing, for each $(\gamma, \kappa, \lambda) \in Y^* \times Z^* \times T^*$, we consider the extended real number $\mathbf{r}_{(\kappa,\lambda)}^\gamma$ and the function $\mathbb{F}_\kappa^\gamma : X \to \overline{\mathbb{R}}$ such that

$$\mathbf{r}_{(\kappa,\lambda)}^\gamma = f^*(\gamma) + (\lambda H)^*(\kappa), \qquad \mathbb{F}_\kappa^\gamma(x) = \gamma F(x) + g(x) + \kappa K(x),\ \forall x \in X.$$

The following lemma will be useful in the proofs in the sequel.

Lemma 2. *For any $(x^*, \gamma, \kappa, \lambda) \in X^* \times Y^* \times Z^* \times T^*$ and $x \in X$, it holds*

$$\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*) + \mathbf{r}_{(\kappa,\lambda)}^\gamma \geq \langle x^*, x \rangle - (f \circ F)(x) - g(x) - \lambda (H \circ K)(x).$$

Proof. Take arbitrarily $(x^*, \gamma, \kappa, \lambda) \in X^* \times Y^* \times Z^* \times T^*$ and $x \in X$. By the definition of $\mathbb{F}_\kappa^\gamma$ and the definition of conjugate function, one has

$$\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*) = \sup_{x' \in X}\left[\langle x^*, x' \rangle - \mathbb{F}_\kappa^\gamma(x')\right] \geq \langle x^*, x \rangle - \gamma F(x) - g(x) - \kappa K(x)$$

and

$$\mathbf{r}_{(\kappa,\lambda)}^\gamma = \sup_{y' \in Y}[\langle \gamma, y' \rangle - f(y')] + \sup_{z' \in Z}[\langle \kappa, z' \rangle - (\lambda H)(z')]$$
$$\geq \langle \gamma, y \rangle - f(y) + \langle \kappa, z \rangle - (\lambda H)(z),\ \forall (y, z) \in Y \times Z.$$

We thus get

$$\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*) + \mathbf{r}_{(\kappa,\lambda)}^\gamma$$
$$\geq \langle x^*, x \rangle - \gamma F(x) - g(x) - (\kappa K)(x) + \langle \gamma, y \rangle - f(y) + \langle \kappa, z \rangle$$
$$- (\lambda H)(z),\ \forall (y, z) \in Y \times Z.$$

Letting $y = F(x)$ and $z = K(x)$, we obtain the desired inequality. □

3.1. Characterizations of generalized composite convex Farkas-type results

In this subsection, we assume that $g \in \Gamma(X)$, $f \in \Gamma(Y)$, $\gamma F \in \Gamma(X)$ for all $\gamma \in \operatorname{dom} f^*$, $\lambda H \in \Gamma(Z)$ for all $\lambda \in T_+^*$, and $\kappa K \in \Gamma(X)$ for all $\kappa \in \operatorname{dom}(\lambda H)^*$ and $\lambda \in T_+^*$. Let us consider the so-called *qualifying set*, which is defined as

$$\mathcal{Q} = \bigcup_{\substack{(\gamma,\lambda) \in \operatorname{dom} f^* \times T_+^* \\ \kappa \in \operatorname{dom}(\lambda H)^*}} \left[\left(0_{X^*}, \mathbf{r}_{(\kappa,\lambda)}^\gamma\right) + \operatorname{epi}\left(\mathbb{F}_\kappa^\gamma\right)^*\right].$$

Lemma 3. *$\mathcal{Q}$ is a convex subset of $X^* \times \mathbb{R}$.*

Proof. Let $(x_i^*, \alpha_i) \in \mathcal{Q}$, $i = 1,2$. Then, there are $(\gamma_i, \lambda_i) \in \operatorname{dom} f^* \times T_+^*$ and $\kappa_i \in \operatorname{dom}(\lambda_i H)^*$ such that $\left(\mathbb{F}_{\kappa_i}^{\gamma_i}\right)^*(x_i^*) + \mathbf{r}_{(\kappa_i,\lambda_i)}^{\gamma_i} \leq \alpha_i$ for all $i = 1,2$. Now, taking $\mu, \nu \in]0,1[$ such that $\mu + \nu = 1$, one has

$$\mu\alpha_1 + \nu\alpha_2 \geq \mu\left(\mathbb{F}_{\kappa_1}^{\gamma_1}\right)^*(x_1^*) + \mu \mathbf{r}_{(\kappa_1,\lambda_1)}^{\gamma_1} + \nu\left(\mathbb{F}_{\kappa_2}^{\gamma_2}\right)^*(x_2^*) + \nu \mathbf{r}_{(\kappa_2,\lambda_2)}^{\gamma_2}.$$

According to the properties of conjugate function and the definition of $\mathbb{F}_\kappa^\gamma$, we get

$$\mu\alpha_1+\nu\alpha_2 \geq \langle \mu x_1^*+\nu x_2^*, x\rangle - \mathbb{F}^{\mu\gamma_1+\nu\gamma_2}_{\mu\kappa_1+\nu\kappa_2}(x) + \langle \mu\gamma_1+\nu\gamma_2, y\rangle - f(y) + \langle \mu\kappa_1+\nu\kappa_2, z\rangle - (\mu\lambda_1+\nu\lambda_2)H(z),$$

for all $(x,y,z)\in X\times Y\times Z$. This leads to

$$\mu\alpha_1+\nu\alpha_2 \geq \left(\mathbb{F}^{\mu\gamma_1+\nu\gamma_2}_{\mu\kappa_1+\nu\kappa_2}\right)^*(\mu x_1^*+\nu x_2^*) + \mathbf{r}^{\mu\gamma_1+\nu\gamma_2}_{(\mu\kappa_1+\nu\kappa_2,\mu\lambda_1+\nu\lambda_2)}. \tag{4}$$

Observe that $(\mu\gamma_1+\nu\gamma_2, \mu\lambda_1+\nu\lambda_2)\in \mathrm{dom} f^*\times T_+^*$ (as $\mathrm{dom} f^*$ and T_+^* are convex sets). Moreover, it is easy to check that $\mu\kappa_1+\nu\kappa_2\in \mathrm{dom}[(\mu\lambda_1+\nu\lambda_2)H]^*$. So, (4) yields $\mu(x_1^*,\alpha_1)+\nu(x_2^*,\alpha_2)\in\mathcal{Q}$, and we are done. □

Theorem 2 ($\mathcal{V}$-stable GCCFR I). *For any $(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$, the following assertions are equivalent:*

(a_1) $(\mathbf{p}^{x^*}_\alpha)$ *holds,*

(b_1) $(x^*,-\alpha)\in\overline{\mathcal{Q}}$.

Proof. We begin by proving

$$\mathrm{epi}\left(f\circ F+g+i_{-T_+}\circ(H\circ K)\right)^* = \overline{\mathcal{Q}}. \tag{5}$$

Let ϕ be the extended real-valued function defined on X^* by

$$\phi(x^*) = \inf_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)^*(x^*) = \inf_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left(\mathbb{F}^\gamma_\kappa\right)^*(x^*) + \mathbf{r}^\gamma_{(\kappa,\lambda)},\ \forall x^*\in X^*.$$

For all $(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*$ and $\kappa\in\mathrm{dom}(\lambda H)^*$, as (C_1) holds, the functions $\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}$, λH and f belong to $\Gamma(X)$. Applying , we deduce that for all $x\in X$,

$$\phi^*(x) = \sup_{x^*\in X^*}\left[\langle x^*,x\rangle - \inf_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)^*(x^*)\right]$$

$$= \sup_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \sup_{u\in X^*}\left[\langle x^*,x\rangle - \left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)^*(x^*)\right]$$

$$= \sup_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)^{**}(x) = \sup_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)(x)$$

$$= \sup_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left[(\gamma F+g+\kappa K)(x) - f^*(\gamma) - (\lambda H)^*(\kappa)\right]$$

$$= g(x) + \sup_{\gamma\in\mathrm{dom} f^*}\left[(\gamma F)(x) - f^*(\gamma)\right] + \sup_{\lambda\in T_+^*}\sup_{\kappa\in\mathrm{dom}(\lambda H)^*}\left[\kappa K(x) - (\lambda H)^*(\kappa)\right]$$

$$= g(x) + f^{**}(F(x)) + \sup_{\lambda\in T_+^*}\{(\lambda H)^{**}(K(x))\} = g(x) + f(F(x)) + \sup_{\lambda\in T_+^*}\{\langle\lambda, H(K(x))\rangle\}$$

$$= f(F(x)) + g(x) + i_{-T_+}\circ(H\circ K)(x).$$

So, $\left(f\circ F+g+i_{-T_+}\circ(H\circ K)\right)^* = \phi^{**}$. It follows from Zalinescu (2002, Theorem 2.3.4) that

$$\mathrm{epi}\left(f\circ F+g+i_{-T_+}\circ(H\circ K)\right)^* = \mathrm{epi}\phi^{**} = \mathrm{epi}\overline{\mathrm{co}\phi} = \overline{\mathrm{coepi}\phi}, \tag{6}$$

where $\overline{\mathrm{co}\phi}$ is the lower semi-continuous convex regularization of ϕ.

On the other hand, according to Bot et al. (2009, p. 931) and Lemma 3, one gets

$$\overline{\mathrm{coepi}\phi} = \overline{\mathrm{coepi}\left(\inf_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)^*\right)} = \overline{\mathrm{co}\left(\bigcup_{\substack{(\gamma,\lambda)\in\mathrm{dom} f^*\times T_+^*\\ \kappa\in\mathrm{dom}(\lambda H)^*}} \mathrm{epi}\left(\mathbb{F}^\gamma_\kappa - \mathbf{r}^\gamma_{(\kappa,\lambda)}\right)^*\right)}$$

$$= \overline{\mathrm{co}\mathcal{Q}} = \overline{\mathcal{Q}} \tag{7}$$

Hence, the desired equality, (5), follows from (6) and (7). What is left is to show that (a_1) is equivalent to (b_1) for all $(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$. By the definition of conjugate function, it is clear that for any $(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$, $(\mathbf{p}_\alpha^{x^*})$ is equivalent to $\big(f\circ F+g+i_{-T_+}\circ(H\circ K)\big)^*(x^*)\le-\alpha$, or equivalently

$$(x^*,-\alpha)\in\operatorname{epi}\big(f\circ F+g+i_{-T_+}\circ(H\circ K)\big)^*. \tag{8}$$

It follows from (5) that (8) is exactly (b_1), and the proof is complete. □

Theorem 3 ($\mathcal{V}$-stable GCCFR II). *Assume that Q is closed regarding the set $\mathcal{V}\times\mathbb{R}$. Then, for any $(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$, the following assertions are equivalent:*

(a_2) $(\mathbf{p}_\alpha^{x^*})$ *holds,*

(b_2) $\exists(\gamma,\lambda)\in\operatorname{dom}f^*\times T_+^*,\ \exists\kappa\in\operatorname{dom}(\lambda H)^*\colon \mathbb{F}_\kappa^\gamma(x)-\mathbf{r}_{(\kappa,\lambda)}^\gamma-\langle x^*,x\rangle\ge\alpha,\ \forall x\in X.$

Proof. By Theorem 2, for any $(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$,

$$(a_2)\quad\Leftrightarrow\quad(x^*,-\alpha)\in\overline{Q}. \tag{9}$$

As Q is closed regarding the set $\mathcal{V}\times\mathbb{R}$ we have

$$(x^*,-\alpha)\in\overline{Q}\quad\Leftrightarrow\quad(x^*,-\alpha)\in Q. \tag{10}$$

It is clear that

$$(x^*,-\alpha)\in Q\Leftrightarrow\exists(\gamma,\lambda)\in\operatorname{dom}f^*\times T_+^*,\ \exists\kappa\in\operatorname{dom}(\lambda H)^*\colon(x^*,-\alpha-\mathbf{r}_{(\kappa,\lambda)}^\gamma)\in\operatorname{epi}\big(\mathbb{F}_\kappa^\gamma\big)^*.$$

According to the definition of epigraph, one gets

$$(x^*,-\alpha)\in Q\Leftrightarrow\exists(\gamma,\lambda)\in\operatorname{dom}f^*\times T_+^*,\ \exists\kappa\in\operatorname{dom}(\lambda H)^*\colon-\alpha-\mathbf{r}_{(\kappa,\lambda)}^\gamma\ge\big(\mathbb{F}_\kappa^\gamma\big)^*(x^*).$$

From the definition of conjugate function, we obtain

$$(x^*,-\alpha)\in Q\quad\Leftrightarrow\quad(b_2). \tag{11}$$

The result now follows from (9), (10) and (11). The proof is complete. □

To shorten notation, we let $(\mathbf{q}_\alpha^{x^*})$ stand for (b_2) in Theorem 3 for all $(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$, i.e.,

$$(\mathbf{q}_\alpha^{x^*})\quad\exists(\gamma,\lambda)\in\operatorname{dom}f^*\times T_+^*,\ \exists\kappa\in\operatorname{dom}(\lambda H)^*\colon\mathbb{F}_\kappa^\gamma(x)-\mathbf{r}_{(\kappa,\lambda)}^\gamma-\langle x^*,x\rangle\ge\alpha,\quad\forall x\in X.$$

In the case when $\mathcal{V}=\{0_{X^*}\}$, $(\mathbf{q}_\alpha^{x^*})$ collapses to

$$(\mathbf{q}_\alpha)\quad\exists(\gamma,\lambda)\in\operatorname{dom}f^*\times T_+^*,\ \exists\kappa\in\operatorname{dom}(\lambda H)^*\colon\mathbb{F}_\kappa^\gamma(x)-\mathbf{r}_{(\kappa,\lambda)}^\gamma\ge\alpha,\quad\forall x\in X.$$

Theorem 4 (Characterizations of $\mathcal{V}$-stable GCCFR). *The following statements are equivalent:*

$(\mathbf{a_3})$ *Q is closed regarding the set $\mathcal{V}\times\mathbb{R}$.*

$(\mathbf{b_3})$ $\forall(x^*,\alpha)\in\mathcal{V}\times\mathbb{R}$, $[(\mathbf{p}_\alpha^{x^*})\Leftrightarrow(\mathbf{q}_\alpha^{x^*})]$.

Proof. $[(\mathbf{a_3})\Rightarrow(\mathbf{b_3})]$ Assume that $(\mathbf{a_3})$ holds. It follows from Theorem 3 that $(\mathbf{b_3})$ is true.

$[(\mathbf{b_3})\Rightarrow(\mathbf{a_3})]$ Assume that $(\mathbf{b_3})$ holds. It is clear that $Q\subset\overline{Q}$. So, we only need to show that $\overline{Q}\cap(\mathcal{V}\times\mathbb{R})\subset Q\cap(\mathcal{V}\times\mathbb{R})$. To do this, take arbitrarily $(x^*,\alpha)\in\overline{Q}\cap(\mathcal{V}\times\mathbb{R})$, which implies that $(x^*,\alpha)\in\overline{Q}$. It follows from this and Theorem 2 that

$$(x^*,\alpha)\in\overline{Q}\quad\Leftrightarrow\quad(\mathbf{p}_{-\alpha}^{x^*}). \tag{12}$$

Since $(\mathbf{b_3})$ holds, we get from (12) that

$$(x^*,\alpha)\in\overline{Q}\quad\Leftrightarrow\quad(\mathbf{q}_{-\alpha}^{x^*}). \tag{13}$$

This, together with (11), gives

$$(x^*,\alpha)\in\overline{Q}\quad\Leftrightarrow\quad(x^*,\alpha)\in Q.$$

The proof is complete. □

In the special case when $\mathcal{V}=\{0_{X^*}\}$ we get some versions of GCCFR from Theorems 2, 3 and 4.

Corollary 1 (GCCFR I). *For any $\alpha\in\mathbb{R}$, the following statements are equivalent:*

$(\mathbf{a_4})$ $(\mathbf{p}_\alpha)$ *holds,*

$(\mathbf{b_4})$ $(0_{X^*},\alpha)\in\overline{Q}$.

Proof. The conclusions follows from Theorem 2 by taking $\mathcal{V}=\{0_{X^*}\}$. □

The following result is a direct consequence of Theorem 3 by taking $\mathcal{V} = \{0_{X^*}\}$.

Corollary 2 (GCCFR II). *Assume that Q is closed regarding the set $\{0_{X^*}\} \times \mathbb{R}$. Then, for any $\alpha \in \mathbb{R}$, the following assertions are equivalent:*

$(\mathbf{a_5})$ $(\mathbf{p}_\alpha)$ *holds,*

$(\mathbf{b_5})$ $(\mathbf{q}_\alpha)$ *holds.*

In the same way, for a special case where $\mathcal{V} = \{0_{X^*}\}$, we get from Theorem 4,

Corollary 3 (Characterizations of GCCFR). *The following statements are equivalent:*

$(\mathbf{a_6})$ *Q is closed regarding the set* $\{0_{X^*}\} \times \mathbb{R}$.

$(\mathbf{b_6})$ $\forall \alpha \in \mathbb{R}$, $[(\mathbf{p}_\alpha) \Leftrightarrow (\mathbf{q}_\alpha)]$.

Remark 1. *The generalized composed convex Farkas-type results developed in this section significantly generalize previous Farkas-type theorems in the literature. In particular, while earlier results such as those in the works of Bot et al. (2006a, 2006b), and Long et al. (2010) focused on either convex systems or DC systems without composed structures, our results apply to composed convex functions with composed constraints under necessary and sufficient conditions. Moreover, the concept of V-stable GCCFR introduced here offers a unified framework that encompasses classical and modern dual characterizations, and the convex set Q provides a flexible tool for verifying such equivalence results. Hence, our contribution covers and extends the scope of earlier Farkas-type systems to a more general setting. The set Q plays a role analogous to the moment cone in classical linear programming, but extended to the non-linear composite setting. Its closedness provides a unifying condition that encompasses various constraint qualifications.*

3.2. A sufficient condition of generalized composite convex Farkas-type results

We maintain the assumptions on the spaces, sets, mappings and functions as in the previous sections. For any $x^* \in X^*$, let us introduce the function $\Phi_{x^*}: X \times Y \times Z \to T^\bullet \times \overline{\mathbb{R}}$ defined by

$$\Phi_{x^*}(x,y,z) = (H(K(x)+z), f(F(x)+y) + g(x) - \langle x^*, x\rangle), \ \forall (x,y,z) \in X \times Y \times Z.$$

Remark 2. *As F is Y_+-convex and K is Z_+-convex, the functions $(x,z) \mapsto K(x)+z$ and $(x,y) \mapsto F(x)+y$ are Y_+- convex and T_+-convex, respectively. Recall that f is convex and Y_+-increasing; H is T_+-convex and $Z_+ - T_+$-increasing; g is convex. So, the functions $(x,z) \mapsto H(K(x)+z)$ and $(x,y) \mapsto f(F(x)+y)+g(x)$ are T_+-convex and $\mathbb{R}_+$- convex, respectively. This entails Φ_{x^*} is $T_+ \times \mathbb{R}_+$-convex for all $x^* \in X^*$.*

The operator $\Pi: X \times Y \times Z \times T \times \mathbb{R} \to Y \times Z \times T \times \mathbb{R}$, defined by $\Pi(x,y,z,t,r) = (y,z,t,r)$ for all $(x,y,z,t,r) \in X \times Y \times Z \times T \times \mathbb{R}$, is the projection operator on $Y \times Z \times T \times \mathbb{R}$. The projection of $\mathrm{epi}_{T\times\mathbb{R}}\Phi_{x^*}$ on $Y \times Z \times T \times \mathbb{R}$ will be denoted by

$$\Omega_{x^*} = \Pi(\mathrm{epi}_{T\times\mathbb{R}}\Phi_{x^*}).$$

It follows from Remark 2 that Ω_{x^*} is a convex subset in $Y \times Z \times T \times \mathbb{R}$.

We now consider the following condition

$$(C_0) \qquad \text{There exists } \hat{x} \in X \text{ such that } \begin{cases} f \text{ is upper semicontinuous at } F(\hat{x}) \\ H \text{ is continuous at } K(\hat{x}), \\ (H \circ K)(\hat{x}) \in -\mathrm{int}T_+. \end{cases}$$

It is worth noting that the condition (C_0) is a sufficient condition of $\mathcal{V}$-stable GCCFR. To see this, we begin by establishing the following lemma, which plays a key role in proving our claim.

Lemma 4. *If (C_0) holds then* $\mathrm{int}\Omega_{x^*} \neq \emptyset$ *for all $x^* \in X^*$.*

Proof. Take arbitrarily $x^* \in X^*$. Since $(H \circ K)(\hat{x}) \in -\mathrm{int}T_+$, there exists a convex neighborhood U_{0_T} of the origin 0_T such that $(H \circ K)(\hat{x}) + U_T^0 \subset -T_+$. By the continuity of H at $K(\hat{x})$, there is a neighborhood U_{0_Z} of 0_Z such that

$$H(K(\hat{x}) + z) \in (H \circ K)(\hat{x}) + \frac{1}{2}U_{0_T}, \quad \forall z \in U_{0_Z}.$$

On the other hand, according to the upper semicontinuity of f at $F(\hat{x})$ and the fact that $(f \circ F)(\hat{x}) < (f \circ F)(\hat{x}) + 1$, we can find a neighborhood U_{0_Y} of 0_Y such that

$$f(F(\hat{x}) + y) < (f \circ F)(\hat{x}) + 1, \quad \forall y \in U_Y^0.$$

Let $\hat{r} = (f \circ F)(\hat{x}) + 1 + g(\hat{x}) - \langle x^*, \hat{x} \rangle$. For any $(y, z, t, r) \in U_{0_Y} \times U_{0_Z} \times \left(-\frac{1}{2}U_{0_T}\right) \times]\hat{r}, \infty[$, one has

$$\begin{cases} -t + H(K(\hat{x}) + z) \in \frac{1}{2}U_{0_T} + (H \circ K)(\hat{x}) + \frac{1}{2}U_{0_T} \subset (H \circ K)(\hat{x}) + U_{0_T} \subset -T_+, \\ r > \hat{r} > f(F(\hat{x}) + y) - g(\hat{x}) - \langle x^*, \hat{x} \rangle, \end{cases}$$

which yields $(\hat{x}, y, z, t, r) \in \mathrm{epi}_{T\times\mathbb{R}}\Phi_{x^*}$, and consequently,

$$U_{0_Y} \times U_{0_Z} \times \left(-\tfrac{1}{2}U_{0_T}\right) \times]\hat{r}, \infty[\subset \Omega_{x^*}. \tag{14}$$

So, $\mathrm{int}\Omega_{x^*} \neq \emptyset$ and we are done. □

Theorem 5 (Stable GCCFR). *Assume that (C_0) holds. Then, for all $(x^*, \alpha) \in X^* \times \mathbb{R}$, $(\mathbf{p}_\alpha^{x^*})$ and $(\mathbf{q}_\alpha^{x^*})$ are equivalent.*

Proof. Take arbitrarily $(x^*, \alpha) \in X^* \times \mathbb{R}$. It is easy to check that $[(\mathbf{p}_\alpha^{x^*}) \Leftarrow (\mathbf{q}_\alpha^{x^*})]$. The proof is completed by showing that the converse implication. Indeed, assume that $(\mathbf{p}_\alpha^{x^*})$ holds, i.e.,

$$(H \circ K)(x) \in -T_+ \ \Rightarrow \ (f \circ F)(x) + g(x) - \langle x^*, x \rangle \geq \alpha, \tag{15}$$

Then, we will prove that $(\mathbf{q}_\alpha^{x^*})$ holds, as well.

• Firstly, we prove that $(0_Y, 0_Z, 0_T, \alpha) \notin \mathrm{int}\Omega_{x^*}$. Suppose, contrary to our claim, that $(0_Y, 0_Z, 0_T, \alpha) \in \mathrm{int}\Omega_{x^*}$. Then, there is $\delta > 0$ satisfying $(0_Y, 0_Z, 0_T, \alpha - \delta) \in \Omega_{x^*}$, which means that there exists $\bar{x} \in X$ such that $(\bar{x}, 0_Y, 0_Z, 0_T, \alpha - \delta) \in \mathrm{epi}_{T\times\mathbb{R}}\Phi_{x^*}$, and hence,

$$\begin{cases} 0_T \geq_T H(K(\bar{x})), \\ \alpha - \delta \geq f(F(\bar{x})) + g(\bar{x}) - \langle x^*, \bar{x} \rangle. \end{cases}$$

This leads to $(H \circ K)(\bar{x}) \in -T_+$ and $f(F(\bar{x})) + g(\bar{x}) - \langle x^*, \bar{x} \rangle < \alpha$, contrary to (15).

• Recall that Ω_{x^*} is a convex subset in $Y \times Z \times T \times \mathbb{R}$ with non-empty interior. So, we now apply the separation theorem () to $\{(0_Y, 0_Z, 0_T, \alpha)\}$ and $\mathrm{int}\Omega_{x^*}$ to obtain the existence of $(\gamma, \kappa, \lambda, \zeta) \in Y^* \times Z^* \times T^* \times \mathbb{R}$ such that

$$\langle \gamma, y \rangle + \langle \kappa, z \rangle + \langle \lambda, t \rangle + \zeta r > \zeta\alpha, \quad \forall (y, z, t, r) \in \mathrm{int}\Omega_{x^*}. \tag{16}$$

Let $\beta := \max\{\alpha, \hat{r}\} + 1 > \hat{r}$, where $\hat{r} = (f \circ F)(\hat{x}) + 1 + g(\hat{x}) - \langle x^*, \hat{x} \rangle$. According to the proof of Lemma 4, $(0_Y, 0_Z, 0_T, \beta) \in U_{0_Y} \times U_{0_Z} \times \left(-\frac{1}{2}U_{0_T}\right) \times]\hat{r}, \infty[$. Combining this with (14), one has $(0_Y, 0_Z, 0_T, \beta) \in \mathrm{int}\Omega_{x^*}$. Substituting $(0_Y, 0_Z, 0_T, \beta)$ into (16) we obtain $\zeta\beta > \zeta\alpha$. This, together with the fact that $\beta > \alpha$, yields $\zeta > 0$. So, by taking $\bar{\gamma} = \frac{\gamma}{\zeta}$, $\bar{\kappa} = \frac{\kappa}{\zeta}$ and $\bar{\lambda} = \frac{\lambda}{\zeta}$, (16) can be rewritten as

$$\langle \bar{\gamma}, y \rangle + \langle \bar{\kappa}, z \rangle + \langle \bar{\lambda}, t \rangle + r > \alpha, \quad \forall (y, z, t, r) \in \mathrm{int}\Omega_{x^*}.$$

Since $\overline{\mathrm{int}\Omega_{x^*}} = \overline{\Omega_{x^*}}$ (Zalinescu (2002, Theorem 1.1.2)), it follows from the above inequality that

$$\langle \bar{\gamma}, y \rangle + \langle \bar{\kappa}, z \rangle + \langle \bar{\lambda}, t \rangle + r \geq \alpha, \quad \forall (y, z, t, r) \in \Omega_{x^*}. \tag{17}$$

• We proceed to show that $\bar{\lambda} \in T_+^*$. Take arbitrarily $t_+ \in T_+$. By the definition of Ω_{x^*} and the fact that $(0_Y, 0_Z, 0_T, \beta) \in \mathrm{int}\Omega_{x^*} \subset \Omega_{x^*}$, we have $(0_Y, 0_Z, \xi t_+, \beta) \in \Omega_{x^*}$ for all $\xi > 0$. Thanks to (17), $\alpha \leq \langle \bar{\lambda}, \xi t_+ \rangle + \beta$, and hence, $\langle \bar{\lambda}, t_+ \rangle \geq \frac{\alpha - \beta}{\xi}$. Letting $\xi \to +\infty$, one gets $\langle \bar{\lambda}, t_+ \rangle \geq 0$, which implies $\bar{\lambda} \in T_+^*$.

• Finally, we claim that $(\mathbf{q}_{\alpha}^{x^*})$ is true. For this purpose, we take arbitrarily $x \in X$. The proof is completed by showing that

$$\mathbb{F}_{-\bar{\kappa}}^{-\bar{\gamma}}(x) - \mathbf{r}_{(-\bar{\kappa},\bar{\lambda})}^{-\bar{\gamma}} - \langle x^*, x\rangle \geq \alpha. \tag{18}$$

Indeed, for any $(u, v) \in Y \times Z$,

$$\Phi_{x^*}(x, u - F(x), v - K(x)) = (H(v), f(u) + g(x) - \langle x^*, x\rangle),$$

which yields $(x, u - F(x), v - K(x), H(v), f(u) + g(x) - \langle x^*, x\rangle) \in \mathrm{epi}_{T\times\mathbb{R}}\Phi_{x^*}$, and hence, $(u - F(x), v - K(x), H(v), f(u) + g(x) - \langle x^*, x\rangle) \in \Omega_{x^*}$. Combining this with (17), one gets

$$\langle\bar{\gamma}, u - F(x)\rangle + \langle\bar{\kappa}, v - K(x)\rangle + \langle\bar{\lambda}, H(v)\rangle + f(u) + g(x) - \langle x^*, x\rangle \geq \alpha,$$

or equivalently,

$$\mathbb{F}_{-\kappa}^{-\gamma}(x) - \langle x^*, x\rangle \geq \alpha + \langle -\bar{\gamma}, u\rangle - f(u) + \langle -\bar{\kappa}, v\rangle + (\bar{\lambda}H)(v). \tag{19}$$

It is worth noting that (18) holds for all $(u, v) \in Y \times Z$. Therefore,

$$\mathbb{F}_{-\bar{\kappa}}^{-\bar{\gamma}}(x) - \langle x^*, x\rangle \geq \alpha + \sup_{u\in Y}[\langle -\bar{\gamma}, u\rangle - f(u)] + \sup_{v\in Z}[\langle -\bar{\kappa}, v\rangle + (\bar{\lambda}H)(v)],$$

which is nothing else but (18). The proof is complete. □

4. DC composed optimization problems with composed constraints

The duality theory developed in this section not only establishes theoretical guarantees but also facilitates the practical development of algorithms based on dual approaches.

We keep maintaining the assumptions on the spaces, sets, mappings and functions as in the previous sections. Consider the convex composed optimization problem with composed constraints and its dual as follows

$$\text{(CCP)} \qquad \inf_{(H\circ K)(x)\in -T_+} [(f \circ F)(x) + g(x)],$$

$$\text{(CCD)} \qquad \sup_{\substack{(\gamma,\lambda)\in \mathrm{dom} f^* \times T_+^* \\ \kappa\in \mathrm{dom}(\lambda H)^*}} \left\{\inf_{x\in X}\mathbb{F}_{\kappa}^{\gamma}(x) - \mathbf{r}_{(\kappa,\lambda)}^{\gamma}\right\}.$$

A vector $x \in X$ is said to be feasible if $(H \circ K)(x) \in -T_+$. If (CCP) has at least one feasible vector then it is called consistent. It is worth noting that the condition (3) ensures that (CCP) consistent. The infimum value of the problem (CCP), denoted by val(CCP), is defined as

$$\mathrm{val(CCP)} := \inf_{(H\circ K)(x)\in -T_+} [(f \circ F)(x) + g(x)].$$

If (CCP) has a feasible vector $\bar{x}$ satisfying $\mathrm{val(CCP)} = (f \circ F)(\bar{x}) + g(\bar{x})$ then (CCP) is said to be solvable and $\bar{x}$ is called a solution of (CCP).

For the problem (CCD), a vector $(\gamma, \kappa, \lambda) \in Y^* \times Z^* \times T^*$ is called feasible if $\gamma \in \mathrm{dom} f^*$, $\lambda \in T_+^*$ and $\kappa \in \mathrm{dom}(\lambda H)^*$. Similarly, we say that the problem (CCD) is consistent if it has a feasible vector. The supremum value of the problem (CCD) will be denoted by val(CCD), where

$$\mathrm{val(CCD)} := \sup_{\substack{(\gamma,\lambda)\in \mathrm{dom} f^* \times T_+^* \\ \kappa\in \mathrm{dom}(\lambda H)^*}} \left\{\inf_{x\in X}\mathbb{F}_{\kappa}^{\gamma}(x) - \mathbf{r}_{(\kappa,\lambda)}^{\gamma}\right\}.$$

If (CCD) has a feasible vector $(\bar{\gamma}, \bar{\kappa}, \bar{\lambda})$ such that $\mathrm{val(CCD)} = \inf_{x\in X}\mathbb{F}_{\kappa}^{\gamma}(x) - \mathbf{r}_{(\kappa,\lambda)}^{\gamma}$ then (CCD) is said to be solvable and $(\bar{\gamma}, \bar{\kappa}, \bar{\lambda})$ is called a solution of (CCD).

For each $x^* \in \mathcal{V}$, we denote the problem (CCP) perturbed by x^* by (CCP_{x^*}) and its Lagrange dual problem by (CCD_{x^*}), i.e.,

$$(\mathrm{CCP}_{x^*}) \qquad \inf_{(H\circ K)(x)\in -T_+} \quad [(f\circ F)(x)+g(x)-\langle x^*,x\rangle]$$

$$(\mathrm{CCD}_{x^*}) \qquad \sup_{\substack{(\gamma,\lambda)\in \mathrm{dom} f^*\times T_+^* \\ \kappa\in \mathrm{dom}(\lambda H)^*}} \left\{\inf_{x\in X}\left[\mathbb{F}_\kappa^\gamma(x)-\langle x^*,x\rangle\right]-\mathbf{r}_{(\kappa,\lambda)}^\gamma\right\}.$$

It is easy to see that the dual problem (CCD_{x^*}) can be rewritten as

$$\sup_{\substack{(\gamma,\lambda)\in \mathrm{dom} f^*\times T_+^* \\ \kappa\in \mathrm{dom}(\lambda H)^*}} \left[-\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*)-\mathbf{r}_{(\kappa,\lambda)}^\gamma\right].$$

Similarly, let us denote by $\mathrm{val}(\mathrm{CCP}_{x^*})$ and $\mathrm{val}(\mathrm{CCD}_{x^*})$ the optimal values of (CCP_{x^*}) and (CCD_{x^*}), respectively.

We now consider the weak duality for the problem (CCP_{x^*}) and its dual.

Theorem 6 (Weak duality). *It holds* $\mathrm{val}(\mathrm{CCD}_{x^*})\le \mathrm{val}(\mathrm{CCP}_{x^*})$ *for all* $x^*\in\mathcal{V}$. *In particular,* $\mathrm{val}(\mathrm{CCD})\le \mathrm{val}(\mathrm{CCP})$.

Proof. Fix $x^*\in\mathcal{V}$. Take arbitrarily $\gamma\in\mathrm{dom} f^*$, $\lambda\in T_+^*$, $\kappa\in\mathrm{dom}(\lambda H)^*$, and $x\in X$ such that $(H\circ K)(x)\in -T_+$. Then, by Lemma 2, one has

$$-\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*)-\mathbf{r}_{(\kappa,\lambda)}^\gamma \le (f\circ F)(x)+g(x)+\lambda(H\circ K)(x)-\langle x^*,x\rangle. \qquad (20)$$

Since $\lambda\in T_+^*$ and $(H\circ K)(x)\in -T_+$, it follows that $\langle \lambda H, K(x)\rangle\le 0$. This together with (20) gives

$$-\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*)-\mathbf{r}_{(\kappa,\lambda)}^\gamma \le (f\circ F)(x)+g(x)-\langle x^*,x\rangle,$$

and hence, $\mathrm{val}(\mathrm{CCD}_{x^*})\le \mathrm{val}(\mathrm{CCP}_{x^*})$. The rest of the proof runs as before. The proof is complete. □

We say that "strong duality for the pair (CCP)-(CCD)" if $\mathrm{val}(\mathrm{CCP})=\mathrm{val}(\mathrm{CCD})$ and the dual problem (CCD) has a solution. We call "$\mathcal{V}$-stable strong duality for the pair (CCP)-(CCD)" if strong duality for the pair (CCP_{x^*})-(CCD_{x^*}) holds for all $x^*\in\mathcal{V}$. If X^*-stable strong duality holds for the pair (CCP)-(CCD), we just say that "stable strong duality holds for the pair (CCP)-(CCD)". In the case $\mathcal{V}=\{0_{X^*}\}$, $\mathcal{V}$-stable strong duality for the pair (CCP)-(CCD) becomes strong duality for the pair (CCP)-(CCD).

We next establish the results about $\mathcal{V}$-stable strong duality for the pair (CCP)-(CCD).

Theorem 7 (Stable strong duality I). *Let* $\emptyset\neq\mathcal{V}\subset X^*$. *Assume that for any* $x^*\in\mathcal{V}$ *the problem* (CCP_{x^*}) *and its dual are both consistent. Then, the following statements are equivalent to each other:*

$(\mathbf{c_1})$ *$\mathcal{Q}$ is closed regarding the set* $\mathcal{V}\times\mathbb{R}$,

$(\mathbf{d_1})$ *$\mathcal{V}$-stable strong duality holds for the pair* (CCP)-(CCD).

Proof. $[(\mathbf{c_1})\Rightarrow(\mathbf{d_1})]$ Assume that $(\mathbf{c_1})$ holds. We claim that $\mathcal{V}$-stable strong duality holds for the pair (CCP)-(CCD). To do this, take arbitrarily $x^*\in\mathcal{V}$ and prove that strong duality holds for pair (CCP_{x^*})-(CCD_{x^*}). Indeed, according to the assumptions, $\mathrm{val}(\mathrm{CCP}_{x^*})<+\infty$. If $\mathrm{val}(\mathrm{CCP}_{x^*})=-\infty$ then by the weak duality one has $\mathrm{val}(\mathrm{CCD}_{x^*})=\mathrm{val}(\mathrm{CCP}_{x^*})=-\infty$, in this case, any pair $(\gamma,\lambda,\kappa)\in\mathrm{dom} f^*\times T_+^*\times\mathrm{dom}(\lambda H)^*$ is a solution of (CCD_{x^*}). We now can assume that $\alpha:=\mathrm{val}(\mathrm{CCP}_{x^*})\in\mathbb{R}$. By the definition of the optimal value of (CCP_{x^*}), we have the statement $(\mathbf{p}_\alpha(x^*))$ is true. Since $(\mathbf{c_1})$ holds, it follows from Theorem 4 that $(\mathbf{q}_\alpha(x^*))$ is true, i.e, there exist $\bar\gamma\in\mathrm{dom} f^*$, $\bar\lambda\in T_+^*$ and $\bar\kappa\in\mathrm{dom}(\lambda H)^*$ such that

$$\mathbb{F}_{\bar\kappa}^{\bar\gamma}(x)-\mathbf{r}_{(\bar\kappa,\bar\lambda)}^{\bar\gamma}-\langle x^*,x\rangle\ge\alpha=\mathrm{val}(\mathrm{CCP}_{x^*}),\quad \forall x\in X,$$

and hence, by the definition of $\mathrm{val}(\mathrm{CCD}_{x^*})$,

$$\mathrm{val}(\mathrm{CCD}_{x^*})\ge\inf_{x\in X}\left[\mathbb{F}_{\bar\kappa}^{\bar\gamma}(x)-\langle x^*,x\rangle\right]-\mathbf{r}_{(\bar\kappa,\bar\lambda)}^{\bar\gamma}\ge\mathrm{val}(\mathrm{CCP}_{x^*}).$$

This, together with the weak duality, yields

$$\mathrm{val}(\mathrm{CCD}_{x^*}) = \inf_{x\in X}\left[\mathbb{F}_{\bar{\kappa}}^{\bar{\gamma}}(x) - \langle x^*, x\rangle\right] - \mathbf{r}_{(\bar{\kappa},\bar{\lambda})}^{\bar{\gamma}} = \mathrm{val}(\mathrm{CCP}_{x^*}).$$

So, $\mathrm{val}(\mathrm{CCD}_{x^*}) = \mathrm{val}(\mathrm{CCP}_{x^*})$ and $(\bar{\gamma}, \bar{\kappa}, \bar{\lambda})$ is a solution of (CCD_{x^*}), which means that $(\mathbf{d_1})$ holds.

$[(\mathbf{d_1}) \Rightarrow (\mathbf{c_1})]$ Assume that $(\mathbf{d_1})$ holds, i.e., $\mathcal{V}$-stable strong duality holds for the pair (CCP)-(CCD). It is clear that $\mathcal{Q} \subset \overline{\mathcal{Q}}$. We only need to show that $\overline{\mathcal{Q}} \cap (\mathcal{V}\times\mathbb{R}) \subset \mathcal{Q} \cap (\mathcal{V}\times\mathbb{R})$. Indeed, take arbitrarily $(x^*, \alpha) \in \overline{\mathcal{Q}} \cap (\mathcal{V}\times\mathbb{R})$. It follows from Theorem 2 that $(\mathbf{p}_{-\alpha}^{x^*})$ holds, which entails

$$-\alpha \le \mathrm{val}(\mathrm{CCP}_{x^*}). \tag{21}$$

As $\mathcal{V}$-stable strong duality holds for the pair (CCP)-(CCD), there is a solution $(\bar{\gamma}, \bar{\kappa}, \bar{\lambda})$ of (CCD_{x^*}) such that

$$\mathrm{val}(\mathrm{CCP}_{x^*}) = -\left(\mathbb{F}_{\bar{\kappa}}^{\bar{\gamma}}\right)^*(x^*) - \mathbf{r}_{(\bar{\kappa},\bar{\lambda})}^{\bar{\gamma}}. \tag{22}$$

Combining (21) with (22), one gets $\alpha \ge \left(\mathbb{F}_{\bar{\kappa}}^{\bar{\gamma}}\right)^*(x^*) + \mathbf{r}_{(\bar{\kappa},\bar{\lambda})}^{\bar{\gamma}}$, which yields $(x^*, \alpha) \in \mathcal{Q}$. So, $\mathcal{Q}$ is closed regarding the set $\mathcal{V}\times\mathbb{R}$, and we are done. □

We now consider the stable strong duality holds for the pair (CCP)-(CCD) which is a consequence of Theorem 7.

Corollary 4 (Stable strong duality II). *Assume that for any $x^* \in X^*$ the problem* (CCP_{x^*}) *and its dual are both consistent. Then, the next two statements are equivalent to each other:*

$(\mathbf{c_2})$ *$\mathcal{Q}$ is w^*-closed in $X^* \times \mathbb{R}$,*

$(\mathbf{d_2})$ *Stable strong duality holds for the pair* (CCP)-(CCD).

Proof. It is a direct consequence of Theorem 7 by taking $\mathcal{V} = X^*$. □

As a consequence of Theorem 7, we now provide the strong duality for the pair (CCP)-(CCD).

Corollary 5 (Strong duality). *Assume that the problem* (CCP) *and its dual are both consistent. Then, the next two statements are equivalent to each other:*

$(\mathbf{c_3})$ *$\mathcal{Q}$ is closed regarding the set $\{0_{X^*}\}\times\mathbb{R}$,*

$(\mathbf{d_3})$ *Strong duality holds for the pair* (CCP)-(CCD).

Proof. The conclusion follows from Theorem 7 by taking $\mathcal{V} = \{0_{X^*}\}$. □

Corollary 6 (Stable strong duality III). *Assume that (C_0) holds. Then, stable strong duality for the pair* (CCP)-(CCD) *holds.*

Proof. The conclusion follows from Theorems 7 and 5. □

Remark 3. *It is worth mentioning that the results established on the stable strong duality and the strong duality for the pair* (CCP)-(CCD) *are new and extend the ones in the works of Bot (2010), Bot et al. (2006a), Jeyakumar et al. (2009), Long et al. (2010) and the references therein. The stable strong duality results obtained in this section substantially improve upon previous duality results for convex and DC programs. Unlike classical approaches that rely on strong constraint qualifications such as Slater-type conditions, our results employ the closedness of a set $\mathcal{Q}$ defined via conjugate functions, allowing duality to be achieved under significantly weaker assumptions. This approach not only unifies the Fenchel-Lagrange dual framework for composed structures but also extends strong duality to the stable setting across perturbations.*

5. Applications

5.1. Duality for the DC convex composed optimization problems with composed constraints

In this section, we study a dual problem for the convex composite problem involving DC functions (difference convex functions) with composed constraints, called *Toland–Fenchel–Lagrange* dual problem (Toland, 1978). All the assumptions on the spaces, sets, mappings and functions are as in the previous sections. Let $h: X \to \overline{\mathbb{R}}$ be a proper, convex and lsc function. Consider the following problem:

$$\text{(DCP)} \qquad \inf_{(H\circ K)(x)\in -T_+} [(f\circ F)(x) + g(x) - h(x)].$$

The stable strong duality for the problem (DCP) can be derived from Corollary 4.

Theorem 10. *Assume that the hypotheses of Corollary 4 hold. Assume further that Q is w^*-closed in $X^* \times \mathbb{R}$. Then, the Toland - Fenchel - Lagrange duality for the problem (DCP) holds, i.e.,*

$$\text{val(DCP)} = \inf_{x^*\in X^*} \max_{\substack{(\gamma,\lambda)\in \operatorname{dom} f^*\times T_+^*,\\ \kappa\in\operatorname{dom}(\lambda H)^*}} \left\{h^*(x^*) - \left(\mathbb{F}_\kappa^\gamma\right)^*(x^*) - \mathbf{r}_{(\kappa,\lambda)}^\gamma\right\}.$$

Proof. According to (23) and Theorem *1*, one has

$$\begin{aligned}\text{val(DCP)} &= \inf_{x\in X}\left[(f\circ F)(x) + g(x) + i_{-T_+}(H\circ K)(x) - h(x)\right]\\ &= \inf_{x^*\in X^*}\{h^*(x^*) - \mathcal{G}^*(x^*)\} \qquad (23)\\ &= \inf_{x^*\in X^*}\{h^*(x^*) + \text{val}(\text{CCP}_{x^*})\}.\end{aligned}$$

Since Q is w^*-closed in $X^* \times \mathbb{R}$, it follows from Corollary *4* that for any $x^* \in X^*$,

$$\text{val}(\text{CCP}_{x^*}) = \max_{\substack{\gamma\in\operatorname{dom} f^*,\ \lambda\in T_+^*,\\ \kappa\in\operatorname{dom}(\lambda H)^*}} \left\{-\left(\mathbb{F}_\kappa^\gamma\right)^*(x^*) - \mathbf{r}_{(\kappa,\lambda)}^\gamma\right\}. \qquad (24)$$

The conclusion comes from (23) and (24), which completes the proof. □

Theorem 11. *Assume that the assumptions of Corollary 6 are true. Then, the Toland - Fenchel - Lagrange duality for the problem (DCP) holds.*

Proof. Similar arguments apply to Corollary *6* by taking $\mathcal{V} = X^*$. □

Remark 5. *It is worth noting that the results on Toland - Fenchel - Lagrange duality for the (DCP) cover some recent ones in the paper of Dinh et al. (2010) and generalize many other results of this type in the literature.*

5.2. Special cases

In this section we consider some special cases of the problem (DCP). Let $C \subset X$, $F_i: X \to \overline{\mathbb{R}}$ for all $i = 1, \dots, k$. We maintain the assumptions on the function h, the mappings H, K as in Subsection *5.1.* Consider the problem with composited constraints of the form

$$\text{(DCMP)} \qquad \inf_{x\in C,\ (H\circ K)(x)\in -T_+} \left\{\max_{i=1,\dots,k} F_i(x) - h(x)\right\}.$$

Set $\Delta_k := \{\gamma = (\gamma_1, \dots, \gamma_k) \in \mathbb{R}_+^k : \sum_{i=1}^k \gamma_i = 1\}$, and

$$\mathcal{R}^1 := \bigcup_{\substack{(\gamma,\lambda)\in\Delta_k\times T_+^*\\ \kappa\in\operatorname{dom}(\lambda H)^*}} \left[(0_{X^*}, (\lambda H)^*(\kappa)) + \operatorname{epi}\left(\sum_{i=1}^k \gamma_i F_i + i_C + \kappa K\right)^*\right].$$

Corollary 8. *Assume that $\mathcal{R}^1$ is weak*-closed in $X^* \times \mathbb{R}$. Then, the Toland - Fenchel - Lagrange duality for the (DCMP) problem holds, i.e.,*

$$\text{val(DCMP)} = \inf_{x^*\in X^*}\left\{\max_{\substack{(\gamma,\lambda)\in\Delta_k\times T_+^*,\\ \kappa\in\operatorname{dom}(\lambda H)^*}}\left\{h^*(x^*) - \left(\sum_{i=1}^k \gamma_i F_i + i_C + \kappa K\right)^*(x^*) - (\lambda H)^*(\kappa)\right\}\right\}.$$

Proof. This is a direct consequence of Theorem *10* with $Y = \mathbb{R}^k$, $g = i_C$, $f: \mathbb{R}^k \to \overline{\mathbb{R}}$ such that $f(y) = \max\{y_1, y_2, \dots, y_k\}$ for all $y = (y_1, y_2, \dots, y_k) \in \mathbb{R}^k$, and $F: X \to \overline{\mathbb{R}^k}$,

$$F(x) = \begin{cases} (F_1(x), F_2(x), \dots, F_k(x)) & \text{if } x \in \bigcap_{i=1}^{k} \mathrm{dom} F_i \\ \infty_{\mathbb{R}^k} & \text{otherwise.} \end{cases}$$

It is clear that for each $\gamma \in \mathbb{R}^k$, $\gamma F(x) = \sum_{i=1}^{k} \gamma_i F_i(x)$ for all $x \in X$, the conjugate function of f is $f^*: \mathbb{R}^k \to \overline{\mathbb{R}}$ and $f^* = i_{\Delta_k}$ (Zalinescu, 2002, Corollary 2.4.17). Then the problem (DCMP) is just a special case of the composite functions problem involving DC functions of the model (DCP) considered in Theorem *10*. In this setting, the set Q is exactly the set $\mathcal{R}^1$. Thus, applying this theorem, we obtain the conclusion of Corollary *8*, and we are done. □

We now consider the case where $X = \mathbb{R}^n$, $Z = T = \mathbb{R}^m$, $C \subset \mathbb{R}^n$, $H(z) = z$ for all $z \in Z$, and $h(x) = 0$ for all $x \in X$. Then, the problem DCMP now collapses to the $min - max$ problem studied in

$$\text{(MMP)} \quad \inf_{x \in C,\, K(x) \in -T_+} \left\{ \max_{i=1,\dots,k} F_i(x) \right\}.$$

We set

$$\mathcal{R}^2 := \bigcup_{(\gamma,\kappa) \in \Delta_k \times T_+^*} \mathrm{epi} \left(\sum_{i=1}^{k} \gamma_i F_i + i_C + \kappa K \right)^*.$$

Corollary 9. *Assume that $\mathcal{R}^2$ is weak*-closed in $X^* \times \mathbb{R}$. Then, the strong duality for the (DCMP) problem holds, i.e.,*

$$\mathrm{val(MMP)} = \max_{(\gamma,\kappa) \in \Delta_k \times T_+^*} \inf_{x \in C} \left(\sum_{i=1}^{k} \gamma_i F_i(x) + (\kappa K)(x) \right).$$

Proof. The conclusion of this corollary follows from Corollary *8* by taking $X = \mathbb{R}^n$, $Z = T = \mathbb{R}^m$, $C \subset \mathbb{R}^n$, $H(z) = z$ for all $z \in Z$, $h(x) = 0$ for all $x \in X$. □

5.3 Illustrative example in finite dimensions

Consider the composite convex optimization problem (CCP) in $\mathbb{R}^2$ with the following data: $X = Y = Z = T = \mathbb{R}^2$, $T^+ = \mathbb{R}_+^2$; $F(x_1, x_2) = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + 3x_2 \end{pmatrix}$; $f(y_1, y_2) = \frac{1}{2}(y_1^2 + y_2^2)$; $g(x_1, x_2) = \delta_C(x_1, x_2)$ where $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0.5, x_2 \geq 0\}$; $H(z_1, z_2) = \begin{pmatrix} z_1 + 0.5z_2 - 1 \\ 0.2z_1 + z_2 - 0.5 \end{pmatrix}$ and $K(x_1, x_2) = \begin{pmatrix} |x_1| \\ x_2^2 \end{pmatrix}$.

The primal problem (CCP) becomes

$$\begin{aligned} \text{(CCP)} \quad \min_{x \in \mathbb{R}^2} \quad & \frac{1}{2}[(2x_1 - x_2)^2 + (x_1 + 3x_2)^2] \\ \text{subject to} \quad & |x_1| + 0.5x_2^2 - 1 \leq 0, \\ & 0.2|x_1| + x_2^2 - 0.5 \leq 0, \\ & x_1 \geq 0.5, \\ & x_2 \geq 0. \end{aligned}$$

The objective function simplifies to $h(x_1, x_2) = 2.5x_1^2 + x_1 x_2 + 5x_2^2$ with gradient $\nabla h(x_1, x_2) = \begin{pmatrix} 5x_1 + x_2 \\ x_1 + 10x_2 \end{pmatrix}$. Analysis of the objective function on the feasible region boundary reveals the optimal value 0.625, attained at (0.5,0).

We derive the dual problem (CCD) by computing the necessary conjugate functions

- $f^*(\gamma_1, \gamma_2) = \frac{1}{2}(\gamma_1^2 + \gamma_2^2)$ with dom $f^* = \mathbb{R}^2$;
- For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$,

$$(\lambda H)^*(\kappa_1,\kappa_2) = \begin{cases} \lambda_1 + 0.5\lambda_2 & \text{if } \kappa_1 = \lambda_1 + 0.2\lambda_2,\ \kappa_2 = 0.5\lambda_1 + \lambda_2 \\ +\infty & \text{otherwise.} \end{cases}$$

- For $x \in C$ $(x_1 \geq 0.5, x_2 \geq 0)$,

$$\mathbb{F}_\kappa^\gamma(x) = (2\gamma_1 + \gamma_2 + \kappa_1)x_1 + (-\gamma_1 + 3\gamma_2)x_2 + \kappa_2 x_2^2.$$

The dual problem (CCD) becomes

$$(\text{CCD}) \quad \max_{\substack{\gamma_1,\gamma_2 \in \mathbb{R} \\ \lambda_1,\lambda_2 \geq 0}} \left\{ \inf_{\substack{x_1 \geq 0.5 \\ x_2 \geq 0}} [(2\gamma_1 + \gamma_2 + \kappa_1)x_1 + (-\gamma_1 + 3\gamma_2)x_2 + \kappa_2 x_2^2] - \frac{1}{2}(\gamma_1^2 + \gamma_2^2) - (\lambda_1 + 0.5\lambda_2) \right\}$$

$$\text{subject to} \quad \kappa_1 = \lambda_1 + 0.2\lambda_2,\ \kappa_2 = 0.5\lambda_1 + \lambda_2.$$

Observe that

$$\inf_{x_1 \geq 0.5}(2\gamma_1 + \gamma_2 + \kappa_1)x_1 = \begin{cases} \gamma_1 + 0.5\gamma_2 + 0.5\kappa_1 & \text{if } 2\gamma_1 + \gamma_2 + \kappa_1 \geq 0 \\ -\infty & \text{otherwise.} \end{cases}$$

So, to ensure finite dual value, we require $2\gamma_1 + \gamma_2 + \kappa_1 \geq 0$. Substituting $\kappa_1 = \lambda_1 + 0.2\lambda_2$ and $\kappa_2 = 0.5\lambda_1 + \lambda_2$, we obtain the explicit dual problem

$$(\text{CCD}) \quad \max_{\substack{\gamma_1,\gamma_2 \in \mathbb{R} \\ \lambda_1,\lambda_2 \geq 0}} \left\{ \inf_{x_2 \geq 0} [(-\gamma_1 + 3\gamma_2)x_2 + (0.5\lambda_1 + \lambda_2)x_2^2] + (\gamma_1 + 0.5\gamma_2 + 0.5\lambda_1 + 0.1\lambda_2) - \frac{1}{2}(\gamma_1^2 + \gamma_2^2) - (\lambda_1 + 0.5\lambda_2) \right\}$$

$$\text{subject to} \quad 2\gamma_1 + \gamma_2 + \lambda_1 + 0.2\lambda_2 \geq 0.$$

Define

$$\alpha = -\gamma_1 + 3\gamma_2;$$
$$\beta = 0.5\lambda_1 + \lambda_2 \quad (\beta \geq 0 \text{ since } \lambda_1, \lambda_2 \geq 0).$$

The solution to the inner minimization problem $\Phi(\alpha,\beta) = \inf_{x_2 \geq 0}[\alpha x_2 + \beta x_2^2]$ is given by

$$\Phi(\alpha,\beta) = \begin{cases} -\dfrac{\alpha^2}{4\beta} & \beta > 0 \text{ and } \alpha \leq 0 \quad \left(\text{minimum at } x_2^* = -\dfrac{\alpha}{2\beta}\right), \\ 0 & \beta > 0 \text{ and } \alpha > 0 \quad (\text{minimum at } x_2^* = 0), \\ 0 & \beta = 0 \text{ and } \alpha \geq 0 \quad (\text{minimum at } x_2^* = 0), \\ -\infty & \beta = 0 \text{ and } \alpha < 0 \quad (\text{unbounded below}). \end{cases}$$

Case 1: $\beta = 0, \alpha \geq 0$

$\beta = 0$ implies $\lambda_1 = \lambda_2 = 0$. The dual objective simplifies to

$$\Psi(\gamma_1,\gamma_2,0,0) = (\gamma_1 + 0.5\gamma_2) - \tfrac{1}{2}(\gamma_1^2 + \gamma_2^2).$$

In Case 1, the dual objective simplifies to a concave quadratic function in γ_1 and γ_2. This function attains its maximum value of 0.625 at $(\gamma_1,\gamma_2) = (1, 0.5)$. Furthermore, the constraints $\alpha \geq 0$ and $2\gamma_1 + \gamma_2 \geq 0$ are satisfied at this point.

Case 2: $\beta > 0, \alpha \leq 0$

The dual objective becomes

$$\Psi(\gamma_1,\gamma_2,\lambda_1,\lambda_2) = -\frac{(-\gamma_1+3\gamma_2)^2}{4(0.5\lambda_1+\lambda_2)} + \gamma_1 + 0.5\gamma_2 - \tfrac{1}{2}(\gamma_1^2 + \gamma_2^2) - 0.5\lambda_1 - 0.4\lambda_2.$$

The gradient condition $\nabla\Psi = 0$ yields no interior solution. On the boundary $\alpha = 0$ $(\gamma_1 = 3\gamma_2)$, the objective becomes

$$\Psi(3\gamma_2,\gamma_2,\lambda_1,\lambda_2) = 3.5\gamma_2 - 5\gamma_2^2 - 0.5\lambda_1 - 0.4\lambda_2 \leq 0.6125 - 0.5\lambda_1 - 0.4\lambda_2 < 0.6125.$$

Since $\beta > 0$ requires at least one of $\lambda_1, \lambda_2 > 0$, the maximum in this case is strictly less than 0.625.

Therefore, the maximum dual value is achieved in Case 1.

We have established

$$\text{val(CCP)} = \text{val(CCD)} = 0.625.$$

This confirms strong duality between the primal problem (CCP) and its Fenchel-Lagrange dual (CCD), despite the nonlinear transformation $K(x) = (|x_1|, x_2^2)$ in the constraints. The equality of primal and dual optimal values validates the theoretical framework.

This example demonstrates the application of our duality framework to problems involving nonlinear composite mappings. The primal problem contains absolute-value and quadratic terms within the constraint mapping K, whereas the dual problem reduces to a four-variable optimization with a one-dimensional sub-problem.

6. Conclusion

In this paper, we have studied a class of generalized convex optimization problems in which both the objective function and the constraint mappings are represented as compositions of convex (or DC) functions. Motivated by the quest for a unified duality framework encompassing composed convex systems and DC structures, we have established generalized Farkas-type results for systems of inequalities involving composite functions under the necessary and sufficient conditions. These results provide a fundamental tool for deriving stable strong duality and strong duality results for a broad class of primal-dual formulations constructed via conjugate duality principles. The proposed framework unifies and extends existing results in convex and DC programming, and it encompasses a wide range of optimization problems, including DC systems and min–max models. Moreover, it addresses the stability of strong duality under perturbations in the sense of Jeyakumar and Li.

The main contributions of this work are as follows: (i) the derivation of characterizations of Farkas-type results for systems involving convex composite functions with composed constraints; (ii) the formulation of duality theorems that extend the classical Fenchel–Lagrange duality to optimization problems involving composed convex and DC structures.

Acknowledgements: The authors are grateful to the anonymous referees for their careful readings and valuable comments which improved the presentation of the paper.

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